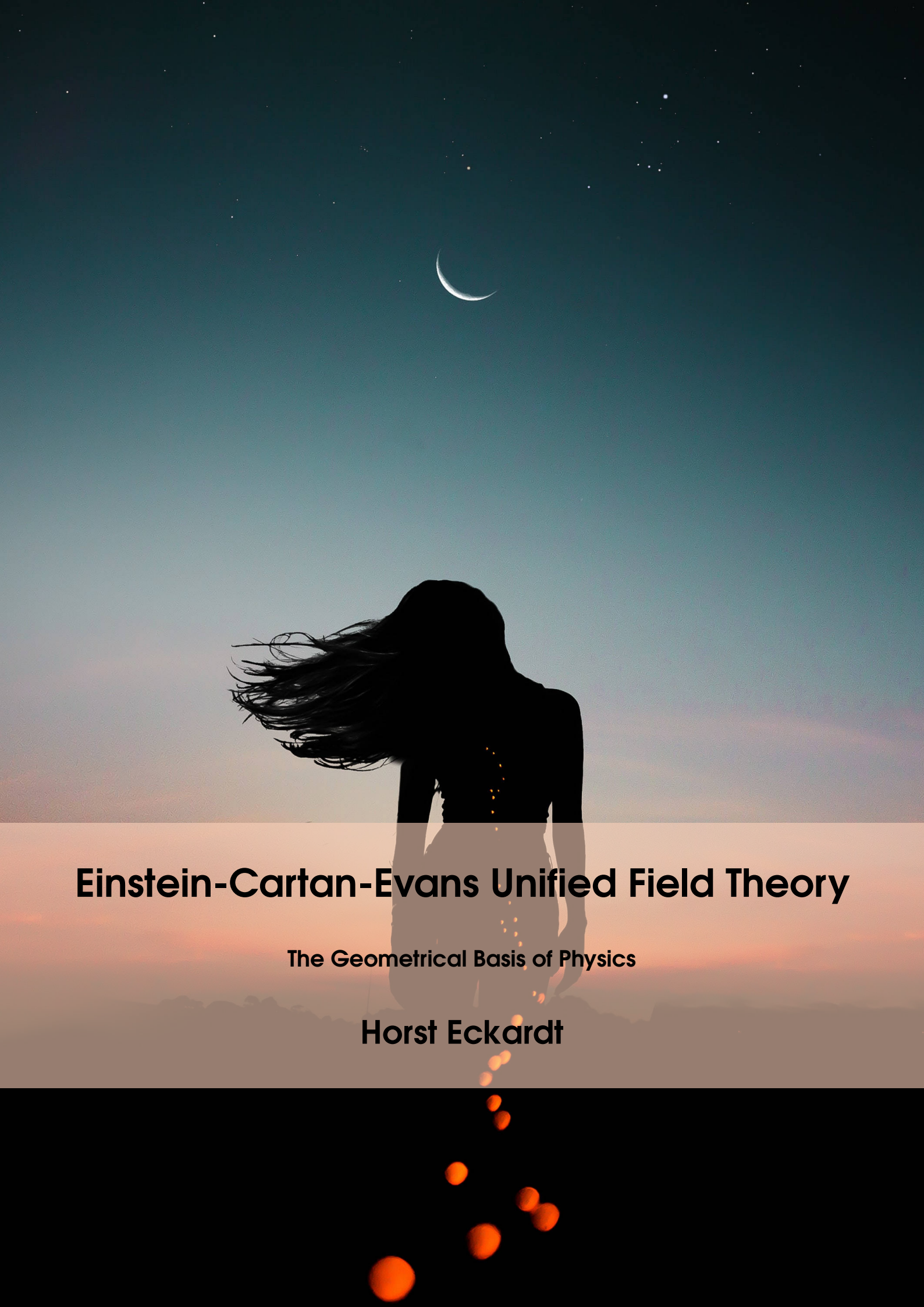

Einstein-Cartan-Evans Unified Field Theory

The Geometrical Basis of Physics

Horst Eckardt

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In Memoriam
Myron W. Evans
(1950 - 2019)

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1. Introduction

Geometry is visible everywhere in daily life. It appears in objects that have been engineered in any form. We are familiar with geometry, since it has been used for centuries (Fig. 1.1). Also, in pure sciences like mathematics and physics, it plays an important role. The mathematical description of geometry consists of the logic elements of geometry itself, for example, the geometric constructions for triangles (Fig. 1.2). This type of logical treatment dates back to the beginning of recorded time, which is assumed be around 3500 B.C., when the first written documents appeared in Mesopotamia. For earlier times, we have to rely on documents of stone, like the pyramids in Egypt, which are probably much older than commonly assumed. The Cheops pyramid has been charted in detail, and correlations have been found to the circumference of the earth, hinting that geometry had had an important role even in the Stone Age. At that time, Europe had a flourishing Celtic culture, from which numerous stone relics exist, and the runes used by the druids were geometric signs.

Ancient philosophy, in particular natural philosophy, culminated in Greece. Pythagoras is said to have been the first founder of mathematics, and we all know the Pythagorean theorem. In Athens, where democracy was born, the “triumvirate” Socrates, Plato and Aristotle founded classical philosophy, starting at about 400 B.C. Their schools were valid for about a thousand years. Euclid, who wrote the pivotal treatise on geometric reasoning, Elements, was a member of the Platonic school.

During medieval times, knowledge from the Roman Empire was preserved by monasteries of the ecclesia and by Arabian philosophers and mathematicians. The Renaissance, which began in Italy in the 14th century and spread to the rest of Europe in the 15th and 16th centuries, was both a rebirth of ancient knowledge, and the beginning of modern empirical natural philosophy. This philosophy is connected with Galileo Galilei, who constituted the method of experimental proofs, and to Johann Kepler, who established our modern heliocentric model of the solar system, first presented as a hypothesis by Nicolaus Copernicus.

Since the 17th century, the mathematical description of physics has made great progress. Isaac Newton published the law of gravitation, which actually goes back to his mentor Robert Hooke. This represented huge progress in natural philosophy, because celestial events could now be predicted mathematically, although this has become completely possible only since the advent of computers.

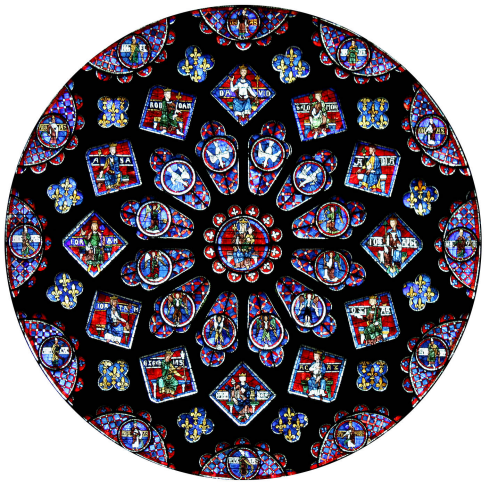

Figure 1.1: Example of geometry: rosette window in the cathedral of Chartres.

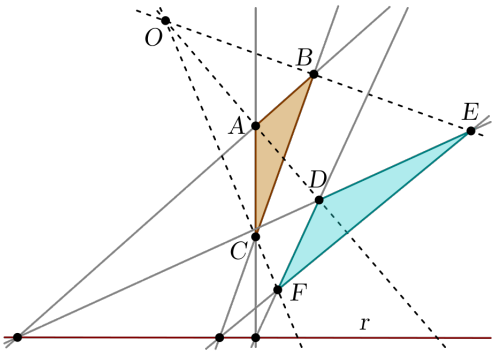

Figure 1.2: Example of geometry: triangles.

The 18th century was the golden age of mechanics. Newton's laws, and their generalizations by Lagrange and Hamilton, paved the way for mathematical physics for the next 300 years. Initially, geometry was used mainly for describing the motion of bodies and particles. The emergence of quantum mechanics in the 20th century extended geometry to the atomic and subatomic realm. For example, the atoms that comprise solids and molecules exhibit a geometrical structure (Fig. 1.3) which is essential for their macroscopic properties. Similar arguments hold for electrodynamics (see, e.g., Fig. 1.4). Faraday's lines of force describe a close-range effect, which was a basis for the geometrical description of electrodynamics, culminating in Maxwell's equations.

The use of geometry changed again at the beginning of the 20th century, when Einstein introduced his theory of general relativity, in which he based physics on non-Euclidean geometry. Gravitation was no longer described by a field imposed externally on space and time, but instead the "spacetime" itself was considered to be an object of description, and altered so that force-free bodies move on a virtually straight line (geodetic line) through space. Spacetime was considered to be curved, and the curving described the laws of gravitation. Along with this interpretation, geometry was considered to be an abstract concept described by numbers and mathematical functions. This approach is known as analytical geometry, and its simplest form uses coordinate systems and vectors.

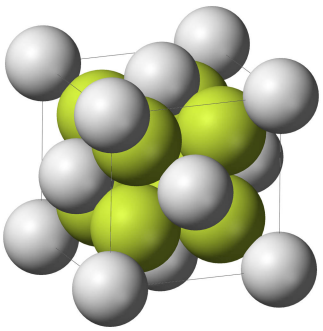

Figure 1.3: Example of geometry: unit cell of Fluorite crystal. White: Calcium, green: Fluorine.

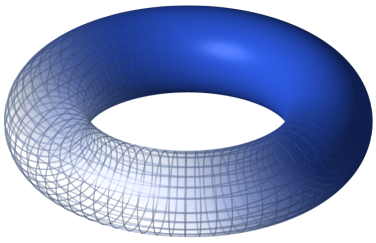

Figure 1.4: Example of geometry: torus.

Einstein's geometrical concept was the first paradigm shift in physics since Newton had introduced his laws of motion, 300 years earlier. Experimental validation of Einstein's general relativity has been rare, and has mainly concerned the solar system, like the deflection of light by the sun and the precession of the orbit of Mercury. In spite of this limitation, the theory was taken as a basis for cosmology, from which the existence of the big bang and dark matter was later extrapolated.

Unfortunately, this approach to cosmology has introduced self-contradictory inconsistencies. For example, the concept that the speed of light is an absolute upper limit is treated like a dogma in contemporary physics, and thus immune to rational argument. However, to explain the first expansive phase of the universe, one has to assume that this happened with an expansion velocity faster than the speed of light. This example is only one of the criticisms of Einstein that have yet to be answered properly and scientifically.

Later, after Einstein's death, the so-called velocity curve of galaxies was observed by astronomers. This means that stars in the outer arms of galaxies do not move according to Newton's law of gravitation, but have a constant velocity. However, Einstein's theory of general relativity is not able to explain this behavior. Both theories (Einstein and Newton) break down in cosmic dimensions. When a theory does not match experimental data, the scientific method requires that the theory be improved or replaced by a better concept. In the case of galactic velocity curves, however, it was "decided" that Einstein is right and that there has to be another reason why stars behave in this way. Dark matter that interacts through gravity and is distributed in a way that accounts for observed orbits was then postulated. Despite an intensive search for dark matter, even on the sub-atomic level, nothing has been found that could interact with ordinary matter through gravity, but not interact with observable electromagnetic radiation, such as light. Sticking with Einstein's theory seems to be a pipe dream, but nobody in the scientific community dares to abandon this non-working theory.

The members of the AIAS institute, Myron Evans at the head, took over the task of developing a new theory of physics that overcomes the problems in Einstein's general relativity. Shortly after the year 2000, Myron Evans developed the "Einstein Cartan Evans theory" (ECE theory [1, 2, 3]) as a replacement, and was even able to unify this with electrodynamics and quantum mechanics. This lead to significant progress in several fields of physics, and the most significant aspects are described in this text book.

ECE theory is based entirely on geometry, as was Einstein's general theory of relativity. Therefore, Einstein is included in the name of this new theoretical approach. Both theories take the geometry of spacetime (three space dimensions, plus one time dimension) as their basis. While Einstein thought that matter curves spacetime and assumed matter to be a "source" of fields, we will see that ECE theory is based entirely on the field concept and does not need to introduce external sources. This idea of sources created a number of difficulties in Einstein's theory.

Another reason for these difficulties is that Einstein made a significant mathematical error in his original theory (1905 to 1915), because all of the necessary information was not yet available. Riemann inferred the metric around 1850, and Christoffel inferred the idea of connection around the 1860s. The idea of curvature was inferred at the beginning of the twentieth century, by Levi Civita, Ricci, Bianchi and colleagues in Pisa. However, torsion was not inferred until the 1920s, by Cartan and his colleagues in Paris.

Therefore, in 1915, when Einstein published his field equations, Riemann geometry contained only curvature, and there was no way of determining that the Christoffel connection must be antisymmetric or at least asymmetric. The arbitrary decision to use a symmetric connection was made into an axiom, and the inferences of Einstein's theory ended up being based on incorrect geometry. Omission of torsion leads to many problems, as has been shown by the AIAS Institute, in great detail [4].

Torsion is a twisting of space, which turns out to be essential and inextricably linked to curvature, because if the torsion is zero then the curvature vanishes [4]. In fact, torsion is even more important than curvature, because the unified laws of gravitation and electrodynamics are basically physical interpretations of twisting, which is formally described by the torsion tensor.

ECE theory unifies physics by deriving all of it directly and deterministically from Cartan geometry, and doing so without using adjustable parameters. Spacetime is completely specified by curvature and torsion, and ECE theory uses these underlying fundamental qualities to derive all of physics from differential geometry, and to predict quantum effects without assuming them (as postulates) from the beginning. It is the first (and only) generally covariant, objective and causal unified field theory.

This book first introduces the mathematics on which ECE theory is based, so that the foundations of the theory can be explained systematically. Mathematical details are kept to a minimum, and explained only as far as is necessary to ensure understanding of the underlying Cartan geometry. This allows the fundamental ECE axioms and theorems to be introduced in a simple and direct manner. The same equations are shown to hold for electrodynamics, gravitation, mechanics and fluid dynamics, which places all of classical physics on common ground. Physics is then extended to the microscopic level by introducing canonical quantization and quantum geometry. The quantum statistics used is classically deterministic. There is no need for renormalization and quantum electrodynamics. All known effects, up to and including the structure of the vacuum, can be explained within the ECE axioms, which are based on Cartan geometry. This is the great advancement that this textbook will explain and clarify.

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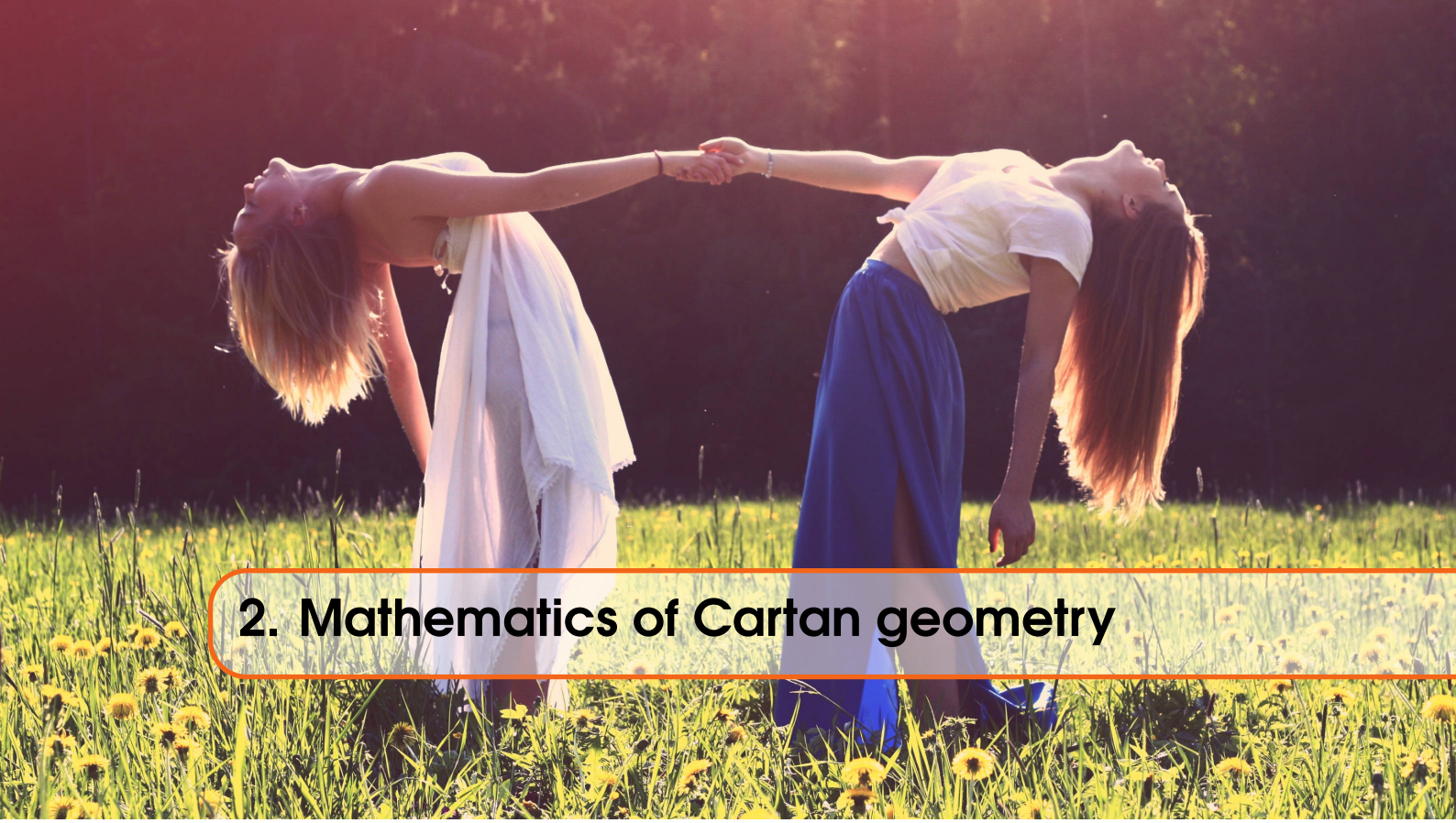

2. Mathematics of Cartan geometry

2.1 Coordinate transformations

Before we can discuss the foundations of non-Cartesian and Cartan geometry on a mathematical level, we need to review the basics of analytical geometry.

2.1.1 Coordinate transformations in linear algebra

To start our discussion of geometry, we first recapitulate some basics of linear algebra. Cartan geometry is a generalization of these concepts, in a sense. Points in space are described by coordinates which are n-tuples for an n-dimensional vector space. The tuple components are numbers and describe how a point in space is reached by putting parts (for example yardsticks) in different directions together. The directions are called base vectors. For a three-dimensional Euclidian space we have the base vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

A point with coordinates (X,Y,Z) is allocated to a vector

$$\mathbf{X} = X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3. \tag{2.2}$$

We have the freedom to choose any base in a vector space, rectangular or not, but when vector analysis is applied to the vector space, it is beneficial to have a rectangular basis. The basis vectors have to be normalized so that this is an orthonormal basis.

A question arises as to what happens when the basis vectors are changed. The position of points in the vector space should be independent of the basis, and we will encounter this fundamental requirement often in Cartan geometry. The coordinates will change when the basis changes. An important part of linear algebra deals with describing this mathematically. Taking the above basis vectors $\mathbf{e}_i$, a new basis $\mathbf{e'}_i$ in an n-dimensional vector space will be a linear combination of the

original basis:

$$\mathbf{e}'_i = \sum_{j=1}^{n} q_{ij}\mathbf{e}_j \tag{2.3}$$

where the coefficients q_{ij} represent a matrix that is commonly called the *transformation matrix*. The above equation can therefore be written as a matrix equation

$$\begin{pmatrix} \mathbf{e}'_1 \\ \vdots \\ \mathbf{e}'_n \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{2.4}$$

with

$$\mathbf{Q} = (q_{ij}) \tag{2.5}$$

and the unit vectors formally arranged in a column vector. $\mathbf{Q}$ must be of rank n and invertible. In Eq. (2.4) the unit vectors can be written with their components as row vectors. Denoting the j-th component of the unit vector $\mathbf{e}_i$ by $(e_i)_j = e_{ij}$, we then can set up a matrix from the unit vectors and write (2.4) in the form

$$\begin{pmatrix} e'_{11} & \dots & e'_{1n} \\ \vdots & & \vdots \\ e'_{n1} & \dots & e'_{nn} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \dots & e_{nn} \end{pmatrix}. \tag{2.6}$$

Then the basis transformation is a matrix multiplication by $\mathbf{Q}$. The matrix for the inverse transformation is obtained by multiplying (2.4) or (2.6) by the inverse matrix $\mathbf{Q}^{-1}$:

$$\begin{pmatrix} e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \dots & e_{nn} \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} e'_{11} & \dots & e'_{1n} \\ \vdots & & \vdots \\ e'_{n1} & \dots & e'_{nn} \end{pmatrix}. \tag{2.7}$$

Multiplying $\mathbf{Q}$ with $\mathbf{Q}^{-1}$ gives the unit matrix which can be expressed by the Kronecker symbol:

$$\mathbf{Q}\,\mathbf{Q}^{-1} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} = (\delta_{ij}). \tag{2.8}$$

■ **Example 2.1** The rotation of bases by an angle ϕ in a two-dimensional vector space can be described by the rotation matrix

$$\mathbf{Q} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{2.9}$$

The basis of unit vectors $(1,0),\ (0,1)$ is then transformed to the new basis vectors

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{2.10}$$

this means

$$\mathbf{e}'_1 = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \mathbf{e}'_2 = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.11}$$

Both basis sets are depicted in Fig. 2.1. ■

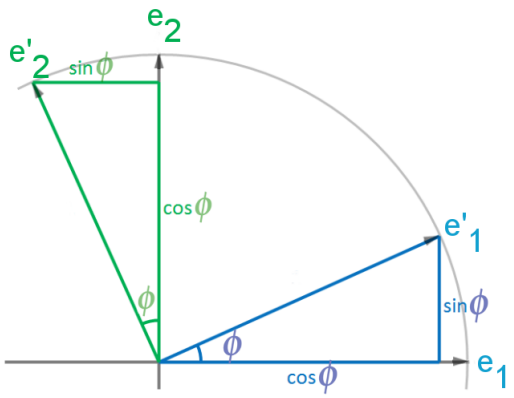

Figure 2.1: Basis vector rotation by an angle ϕ.

Now that we understand the basis transformation, we want to find the transformation law for vectors. The components of vectors in one base, the coordinates, are transformed to the components in another base. From the definition (2.2) a vector with coordinates x_i can be written as

$$\mathbf{X} = \sum_i x_i \mathbf{e}_i \tag{2.12}$$

and may be transformed to a representation in a second basis with coordinates x'_i:

$$\mathbf{X}' = \sum_i x'_i \mathbf{e}'_i. \tag{2.13}$$

Since the vector should remain the same in both bases, we can set $\mathbf{X} = \mathbf{X}'$. Inserting the basis transformations into this relation, one finds that the transformation law of coordinates is

$$\mathbf{X}' = \mathbf{Q}^{-1}\,\mathbf{X} \tag{2.14}$$

or in coordinates:

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{2.15}$$

We notice the important result that the coordinates transform with the inverse matrix compared to the basis vectors and vice versa.

■ **Example 2.2** The transformation matrix of coordinates for the rotation in two dimensions is

$$\mathbf{Q}^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{2.16}$$

This can easily be seen because the reverse rotation is by an angle $-\phi$. Then the sine function reverses sign but the cosine function does not. The vectors on the basis axes are transformed to

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.17}$$

■

Comparing with (2.9), we see that the columns of the transformation matrix $\mathbf{Q}$ represent the coordinates of the transformed unit *vectors*, not the *basis*. In general:

$$\begin{pmatrix} x_{11} & \dots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \dots & x_{nn} \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} \tag{2.18}$$

where $x_{ij} = (x_i)_j$ denotes the j$^{\text{th}}$ component of the transformed unit vector $\mathbf{e}_i$. Please notice that the index scheme for x_{ij} is transposed compared to the usual matrix definition.

2.1.2 General coordinate transformations and coordinate differentials

In the framework of general relativity, coordinate transformations are mappings from one vector space to another. These mappings are multidimensional functions. In the preceding section we restricted ourselves to linear transformations (or mappings), while in general relativity we operate with nonlinear transformations.

Space is described by a four-dimensional manifold, using advanced mathematics. However, in this book we do not develop these concepts in any great extent, but only explain the parts that are required for a basic understanding. The mathematical details can be found in textbooks on general relativity, for example, see [5]-[9].

In this book, we use one time-coordinate plus three space-coordinates for general relativity, with indices numbered from 0 to 3. Such vectors are also called 4-vectors. The functions and maps (later: the tensors) defined on this base space are functions of the coordinates: $f(x_i)$, i=0...3. In particular, coordinate transformations can be described in this form. Let's consider two coordinate systems A and B which describe the same space and are related by a nonlinear transformation. Let X_i be the components of a 4-vector $\mathbf{X}$ in space A and Y_i the components of a 4-vector $\mathbf{Y}$ in space B. The coordinate transformation function $f : \mathbf{X} \to \mathbf{Y}$ then can be expressed as a functional dependence of the components:

$$Y_i = f_i(X_j) = Y_i(X_j) \tag{2.19}$$

for all components i of f and all pairs i, j. In the following, we consider the transformations between a rectangular, orthonormal coordinate system, defined by basis vectors $(1,0,...), (0,1,...)$, etc., and coordinates

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \tag{2.20}$$

and a curvilinear coordinate system with coordinates

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{2.21}$$

The transformation functions from the curvilinear to the cartesian coordinate system may be defined by

$$X_i = X_i(u_j) \tag{2.22}$$

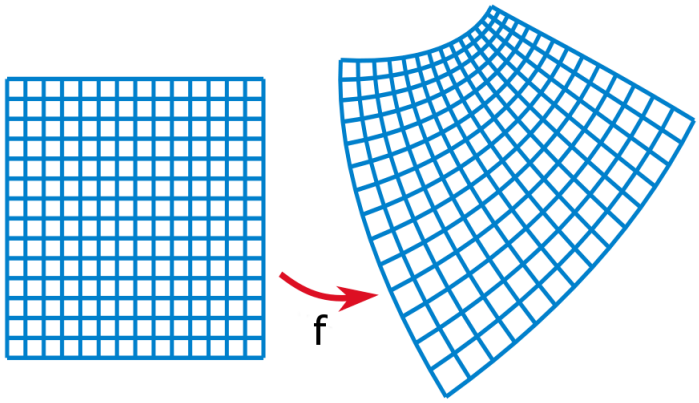

Figure 2.2: Transformation to curvilinear coordinates.

as discussed above. The inverse transformations define the *coordinate functions* of u:

$$u_i = u_i(X_j). \tag{2.23}$$

The functions $u_i =$ constant define *coordinate surfaces*, see Fig. 2.2, for example.

The degree of change in each direction is given by the change of arc length and is expressed by the *scale factors*

$$h_i = \left| \frac{\partial \mathbf{X}}{\partial u_i} \right|. \tag{2.24}$$

The unit vectors in the curvilinear space are computed by

$$\mathbf{e}_i = \frac{1}{h_i} \frac{\partial \mathbf{X}}{\partial u_i}. \tag{2.25}$$

The *tangent vector* of the coordinate curves at each point of space is defined by

$$\nabla u_i = \sum_j \frac{\partial u_i}{\partial X_j} \mathbf{e}_j. \tag{2.26}$$

We require that curvilinear coordinate system be orthonormal at each point of space. This can be assured by the condition that the tangent vectors of the coordinate curves at each point fulfill the requirement

$$\nabla u_i \cdot \nabla u_j = \delta_{ij}. \tag{2.27}$$

The scale factors can alternatively be expressed by the modulus of the tangent vector:

$$h_i = \frac{1}{|\nabla u_i|}. \tag{2.28}$$

■ **Example 2.3** We consider the transformation from cartesian coordinates to spherical coordinates in Euclidean space. The curvilinear coordinates of a point in space are (r, θ, ϕ), where r is the radius, θ the polar angle and ϕ the azimuthal angle, see Fig. 2.3. The cartesian coordinates are (X, Y, Z). The transformation equations from the curvilinear to the rectangular coordinate system are

$$\begin{aligned} X &= r \sin\theta \cos\phi \\ Y &= r \sin\theta \sin\phi \\ X &= r \cos\theta \end{aligned} \tag{2.29}$$

and the inverse transformations are

$$\begin{aligned} u_r &= r = \sqrt{X^2+Y^2+Z^2} \\ u_\theta &= \theta = \arccos \frac{Z}{\sqrt{X^2+Y^2+Z^2}} \\ u_\phi &= \phi = \arctan \frac{Y}{X}. \end{aligned} \tag{2.30}$$

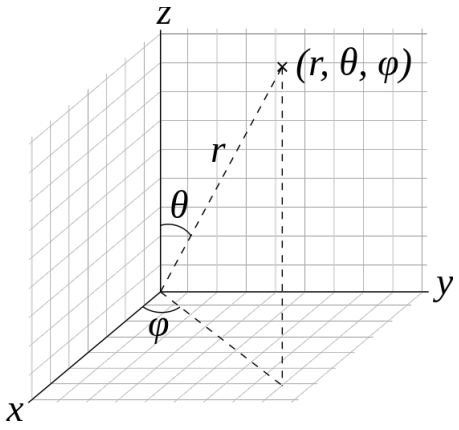

Figure 2.3: Spherical polar coordinates [67].

The vector of scale factors (2.24) is

$$\mathbf{h} = \begin{pmatrix} 1 \\ r \\ r\sin\theta \end{pmatrix} \tag{2.31}$$

and the matrix of column unit vectors (2.25) is

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \begin{pmatrix} \cos\phi\,\sin\theta & \sin\phi\,\sin\theta & \cos\theta \\ \cos\phi\,\cos\theta & \sin\phi\,\cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{2.32}$$

The components of $\mathbf{h}$ have to be positive. The sine function may have positive and negative values, but in spherical coordinates the range of θ is between 0 and π, therefore this function is always positive. ∎

As explained, the coordinate systems are chosen in a way that ensures that the length of vectors is conserved. This must also hold for time-dependent processes. For example, a distance vector changing over time is

$$\Delta\mathbf{X} = \mathbf{v}\Delta t - \mathbf{X}_0 \tag{2.33}$$

where $\mathbf{v}$ is the velocity vector of a mass point and $\mathbf{X}_0$ is an offset. The squared distance is

$$s^2 = v^2\Delta t^2 - (\mathbf{X}_0)^2. \tag{2.34}$$

We notice that a minus sign appears in front of the space part of s^2. This is different from pure "static" Euclidean 3-space, where we have

$$s_E^2 = X^2 + Y^2 + Z^2. \tag{2.35}$$

Now we generalize Eq. (2.34). When the differences in time as well as in space between two points are infinitesimally different, we can write the distance between these points with coordinate differentials:

$$ds^2 = c\,dt^2 - dX^2 - dY^2 - dZ^2 \tag{2.36}$$

where ds is the differential line element. We have added a factor c to the time coordinate t so that all coordinates have the physical dimension of length. In the same way, we can express the line element in another coordinate system, say u coordinates:

$$ds^2 = (du_0)^2 - (du_1)^2 - (du_2)^2 - (du_3)^2. \tag{2.37}$$

So far we have dealt with a Euclidian 3-space, augmented by a time component. More generally, the above equations can be written in the form

$$ds^2 = \sum_{ij} \eta_{ij}\, dx_i\, dx_j \tag{2.38}$$

where η_{ij} represents a matrix of constant coefficients directly leading to the result (2.36) or (2.37):

$$(\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.39}$$

Formally, we can write the coordinates as a 4-column vector

$$(x^\mu) = \begin{pmatrix} ct \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \tag{2.40}$$

where μ runs from 0 to 3 and is written as an upper index. We can do the same for the coordinate differentials:

$$(dx^\mu) = \begin{pmatrix} c\,dt \\ dX^1 \\ dX^2 \\ dX^3 \end{pmatrix}. \tag{2.41}$$

At this point, we should notice that the determinant of the matrix (2.39) is -1. The η matrix is called the *metric* of the space, here the time-extended flat Euclidean space, also called *Minkowski space*. Obviously, the metric is negative definite. Sometimes η is defined with reverse signs but the result is the same. At this point, we enter the realm of special relativity, but we need not deal with Lorentz transformations in this book. Since the spacetime metric is an essential physical quantity in general relativity as well as in ECE theory, we introduce special relativity only under the view point that the line element ds is independent of the coordinate system. Later we will see that this leads to the gamma factor of special relativity. This is the only formalism in common between Einstein's relativity and ECE theory. We will come back to this when physical situations are considered where very high velocities occur. This requires a relativistic treatment (in the sense of special relativity).

2.1.3 Transformations in curved spaces

So far, we have done linear algebra in Euclidean spaces, but now we are extending the concepts of the preceding section to curved spaces. This means that equidistant coordinate values do not describe line elements equal in length. But we should be warned: Using such coordinate systems does not mean that space is "curved" in any way. According to Example 2.1 in the preceding section, a curvilinear coordinate system can perfectly describe a Euclidean "flat" space.

Below, we consider two coordinate systems existing in the same space, denoted by primed and un-primed differentials dx and dx'. According to Eqs. (2.22, 2.23) we have a functional dependence between both coordinates:

$$x^\mu = x^\mu(x'^\nu) \tag{2.42}$$

and

$$x'^\nu = x'^\nu(x^\mu). \tag{2.43}$$

Differentiating these equations gives

$$dx'^\mu = \sum_\nu \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \tag{2.44}$$

$$dx^\mu = \sum_\nu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu. \tag{2.45}$$

To make these equations similar to the transformations in linear algebra (see Section 2.2.2), we define transformation matrices

$$\alpha^\mu_\nu = \frac{\partial x^\mu}{\partial x'^\nu}, \tag{2.46}$$

$$\overline{\alpha}^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{2.47}$$

so that any vector V with components V^μ in one coordinate system can be transformed to a vector V' in the other coordinate system by

$$V'^\mu = \alpha^\mu_\nu \, V^\nu, \tag{2.48}$$

$$V^\mu = \overline{\alpha}^\mu_\nu \, V'^\nu. \tag{2.49}$$

These matrices, however, are not elements of linear algebra but matrix functions, because we are not working with linear transformations. α is the inverse matrix function of $\overline{\alpha}$ and vice versa. This means:

$$\sum_\rho \alpha^\mu_\rho \, \overline{\alpha}^\rho_\nu = \delta^\mu_\nu \tag{2.50}$$

with the Kronecker delta

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}. \tag{2.51}$$

Here we have written α with an upper and lower index intentionally. This allows us to introduce the *Einstein summation convention*: if the same index appears as an upper and a lower index on one side of an equation, this index is summed over. Such an index is also called a *dummy index*. We

will use this feature intensively, when tensors are introduced later. With this convention, which we will use without notice in the future, we can write:

$$\alpha^{\mu}_{\rho}\,\overline{\alpha}^{\rho}_{\nu} = \delta^{\mu}_{\nu}. \tag{2.52}$$

Since space is not necessarily flat, the metrical coefficients of (2.39) are not constant, and non-diagonal terms may appear. This general metric is conventionally called $g_{\mu\nu}$ and defined by the line element as before:

$$ds^2 = g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}. \tag{2.53}$$

For a flat space with cartesian coordinates we have

$$g_{\mu\nu} = \eta_{\mu\nu}. \tag{2.54}$$

■ **Example 2.4** We compute an example for a transformation matrix. Using Example 2.3 (transformation between cartesian coordinates and spherical polar coordinates), we have by (2.29), (2.30):

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta \end{aligned} \tag{2.55}$$

and the inverse transformations

$$\begin{aligned} x'^1 &= r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2} \\ x'^2 &= \theta = \arccos\frac{x^3}{\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}} \\ x'^3 &= \phi = \arctan\frac{x^2}{x^1}. \end{aligned} \tag{2.56}$$

The transformation matrix is according to (2.46):

$$\begin{aligned} \alpha^1_1 &= \frac{\partial x^1}{\partial x'^1} = \frac{\partial}{\partial r}(r\sin\theta\cos\phi) = \sin\theta\cos\phi \\ \alpha^1_2 &= \frac{\partial x^1}{\partial x'^2} = \frac{\partial}{\partial \theta}(r\sin\theta\cos\phi) = r\cos\theta\cos\phi \end{aligned} \tag{2.57}$$

etc. ...

resulting in the 3x3 matrix

$$\alpha = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.58}$$

Obviously, this matrix is not symmetric and even has a zero on the main diagonal. Nonetheless, it is of rank 3 and is invertible, as can be checked. We omit the details here, since the inverse matrix is a bit complicated. The determinant of α is $r^2\sin\theta$, the determinant of the inverse matrix $\overline{\alpha}$ is $1/(r^2\sin\theta)$. By insertion, one can check that

$$\alpha\cdot\overline{\alpha} = \mathbf{1}. \tag{2.59}$$

This example is available as code for the computer algebra system Maxima [46]. ■

■ **Example 2.5** As a further example, we will compute the metric of the coordinate transformation of the previous example (2.4), see computer algebra code [47]. So far, we have no formal method given to do this. The simplest way for Euclidean spaces is the method going back to Gauss. If the metric $\mathbf{g}$ (a matrix) is known for one coordinate system x^μ, the invariant line element of a surface (which is hypothetical in our case) is given by

$$ds^2 = \begin{bmatrix} dx^1 dx^2 dx^3 \end{bmatrix} \mathbf{g} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}. \tag{2.60}$$

The metrical matrix belonging to another coordinate system x'^μ is then computable by

$$\mathbf{g}' = \mathbf{J}^T \, \mathbf{g} \, \mathbf{J} \tag{2.61}$$

where $\mathbf{J}$ is the Jacobian of the coordinate transformation:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\ \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3} \end{bmatrix}. \tag{2.62}$$

Comparing this with Eq. (2.57), we see that the transformation matrix α is identical with the Jacobian, so we can also write:

$$\mathbf{g}' = \alpha^T \, \mathbf{g} \, \alpha. \tag{2.63}$$

The metric of the cartesian coordinates is simply

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.64}$$

and can be inserted into (2.63), together with α from the preceding example. The result is

$$\mathbf{g}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{2.65}$$

for the metric of the spherical coordinates. Written as the line element, this is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{2.66}$$

The metric is symmetric in general, and diagonal in most relevant cases. We will learn other methods of determining the metric in curved spaces during the course of this book. ■

2.2 Tensors

Now that we have explained coordinate transformations and their matrix representations, including the metric, to some extent, we will extend this formalism from vectors to tensors. First, we have to define what a tensor is, and then we can see how they are transformed.

In Section 1.1.3 we introduced the formalism of writing matrices and vectors by indexed quantities, with upper or lower index, where this position was chosen more or less arbitrarily, for example to fulfill the Einstein summation convention. Now let's introduce k-dimensional objects (k ranging from 0 to any integer number) with upper and lower indices of the form

$$T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}. \tag{2.67}$$

T has n upper indices μ_i and m lower indices ν_i with $n+m=k$. It is not required that all upper indices appear first, for example

$$T^{3\ \ 30}_{\ 1} \tag{2.68}$$

is a valid object. The indices μ_i,ν_i represent the coordinate indices for each dimension, ranging from 0 to k-1 by definition. In the above example we have $k=4$, so

$$T^{3\ \ 40}_{\ 5} \tag{2.69}$$

would not be a valid object. For $k=2$ such an object represents a matrix, for $k=1$ a vector and for $k=0$ (without index) a scalar value. A tensor is defined by objects of type (2.67) which adhere to a certain transformation behavior of the upper and lower indices. Given a coordinate transformation α^μ_ρ between two coordinate systems, this transformation has to be applied for each index of a tensor separately. For example, a 2-dimensional tensor T may be transformed to T' by

$$T'^{\mu\nu} = \alpha^\mu_\rho\ \alpha^\nu_\lambda\ T^{\rho\lambda}. \tag{2.70}$$

We further require that for lower indices we use the inverse transformation matrices:

$$T'_{\mu\nu} = \overline{\alpha}^\rho_\mu\ \overline{\alpha}^\lambda_\nu\ T_{\rho\lambda} \tag{2.71}$$

and, consequently, for mixed cases:

$$T'^{\mu}_{\ \ \nu} = \alpha^\mu_\rho\ \overline{\alpha}^\lambda_\nu\ T^\rho_{\ \lambda}\ . \tag{2.72}$$

Please notice that the α matrices are defined by the differentials of the transformation, see Eqs. (2.44, 2.45).

In Section 2.1.1 we have seen that, if α^μ_ρ transforms the basis vectors, then the inverted matrix $\overline{\alpha}^\nu_\lambda$ transforms the coordinates of vectors. Therefore, the upper indices of tensors transform like coordinates, while the lower indices transform like the basis. Upper indices are also called *contravariant indices*, while lower indices are called *covariant indices*. A tensor containing both types of indices is called a *mixed index tensor*.

We conclude this section with the hint that the metric introduced in the previous section is also a tensor. Mathematically, more precisely, we would restrict the tensors then to live in metric spaces, but we won't bother too much with mathematical details in this textbook. The metric $g_{\mu\nu}$ in curved spaces is a symmetric matrix and a tensor of dimension 2. The inner product of two vectors v, w can be written with aid of the metric:

$$s = g_{\mu\nu}\ v^\mu w^\nu. \tag{2.73}$$

In Euclidean space with Cartesian coordinates, g is the unit matrix as demonstrated in Example (2.5). Indices of arbitrary tensors can be moved up and down via the relations

$$T^\mu_{\ \ \nu} = g_{\nu\rho}\ T^{\mu\rho} \tag{2.74}$$

and

$$T^\mu_{\ \ \nu} = g^{\mu\rho}\ T_{\rho\nu} \tag{2.75}$$

where $g^{\mu\rho}$ is the inverse metric:

$$g^{\mu\rho} g_{\nu\rho} = \delta^\mu_{\ \ \nu}\,. \tag{2.76}$$

■ **Example 2.6** We present several tensor operations. Tensors can be multiplied. Then the product has the union set of indices, for example

$$A^{\mu\nu}B_{\rho} = C^{\mu\nu}{}_{\rho}. \tag{2.77}$$

The order of multiplication of A and B plays a role. Therefore, such a product is only meaningful for tensors with a certain symmetry, for example the product tensor of two vectors:

$$v^{\mu}w^{\nu} = C^{\mu\nu}. \tag{2.78}$$

Here C is a symmetric tensor, i.e.

$$C^{\mu\nu} = C^{\nu\mu}. \tag{2.79}$$

Only tensors with the same rank can be added:

$$A^{\mu}{}_{\nu} + B^{\rho}{}_{\sigma} = C^{\alpha}{}_{\beta}. \tag{2.80}$$

The equation

$$A^{\mu}{}_{\nu} + B^{\rho\sigma}{}_{\tau} \stackrel{?}{=} C^{\alpha\beta}{}_{\tau} \tag{2.81}$$

is not compatible with the definition of tensors, and is therefore wrong. For further examples, see [5]. ■

2.3 Base manifold and tangent space

Now that we have seen an overview of the tensor formalism, we will consider the spaces on which these tensors are operating. A tensor can be considered as a function, for example

$$T^{\mu}{}_{\nu} : \mathbb{R}^4 \to \mathbb{R}^2 \tag{2.82}$$

which maps a 4-vector to a two-dimensional tensor field:

$$[ct, X, Y, Z] \to T^{\mu}{}_{\nu}\,(ct, X, Y, Z) \tag{2.83}$$

where the two indices of the tensor indicate that the image map is two-dimensional. We speak of "tensor field" in cases where a continuous argument range is mapped to a continuous image range which is different from the argument set. For example, T could be an electromagnetic field which is defined at each point of 4-space. If the set of arguments is not Euclidean, we require that, at each point of the argument set, a local neighborhood exists, which is homomorphous to an open subset of $\mathbb{R}^n$ where n is the dimension of the argument set. This is then called a *manifold*. Applying multiple tensor functions to a manifold means that several *maps* of the manifold exist. It is further required that the manifold is differentiable because we want to apply the differential calculus later. Assume that a point P is located within the valid local range of two different coordinate systems. Then the manifold is differentiable in P, if the Jacobian of the transformation between both coordinate systems is of rank n, the dimension of the manifold. For definition of scalar products, lengths, angles and volumes we need a metric structure for "measurements", and this requires the existence of a metric tensor. A differentiable manifold with a metric tensor is called *Riemannian manifold*.

■ **Example 2.7** In Fig. 2.4 an example for a 2-dimensional manifold is given: the surface of the earth. The geometry is non-Euclidean. For large triangles on the earth's surface, the sum of angles is different from 180°. A small region is mapped to a flat area where Euclidean geometry is re-established. This can be done for each point of a manifold within a neighborhood, but not globally for the whole manifold. ■

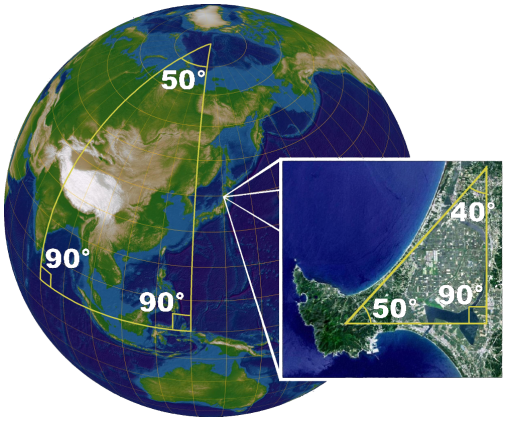

Figure 2.4: 2-dimensional manifold and mapping of a section to a plane segment.

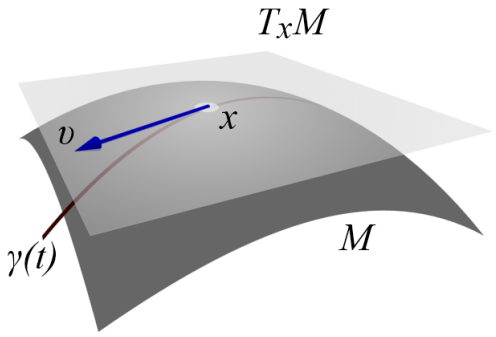

Figure 2.5: Tangential vector v to a 2-dimensional manifold M.

At each point of such a manifold a *tangential space* can be defined. This is a flat $\mathbb{R}^n$ space with the same dimension as the manifold. In Fig. 2.5 an example of a 2-dimensional manifold and tangent space is depicted. The manifold is denoted by M and the tangent space at the point x by T_xM. Such a tangent space (a plane) for example occurs for the motion of mass points along an orbital curve $\gamma(t)$.

The manifold can be covered by points with local neighborhoods and corresponding tangent spaces in each of these points. The set of all tangent spaces is called the *tangent bundle*. Changing the coordinate systems within the manifold means that the mapping from the manifold to the tangential space has to be redefined. A scalar product can be defined in the tangential space by use of the metric of the manifold.

Now we want to make the definition of tangent space independent of the choice of coordinates. The tangent space T_xM at a point x in the manifold can be identified with the space of directional derivative operators along curves through x. The partial derivatives $\frac{\partial}{\partial x^\mu} = \partial_\mu$ represent a suitable basis for the vector space of directional derivatives, which we can therefore safely identify with the tangent space.

Consider two manifolds M and N and a function $F: M \to N$ for a mapping of points of M to points in N. In M and N no differentiation is defined. However, we can define *coordinate charts* from the manifolds to their corresponding tangent spaces. These are the functions denoted by ϕ and ψ in Fig. 2.6. The coordinate charts allow us to construct a map between both tangent spaces:

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n. \tag{2.84}$$

With aid of this construct we can define a partial derivative of f exploiting the indirection via the

tangent spaces. For a point x^μ in $\mathbb{R}^m$ (the mapped point x of M) we define:

$$\frac{\partial f}{\partial x^\mu} := \frac{\partial}{\partial x^\mu}(\psi \circ f \circ \phi^{-1})(x^\mu). \tag{2.85}$$

In many application cases we have a curve in the manifold M described by a parameter λ. This could be the motion of a mass point in dependence of time. Similarly, as above, we can define the derivative of function f according to λ by using the chain rule:

$$\frac{df}{d\lambda} := \frac{dx^\mu}{d\lambda}\partial_\mu f. \tag{2.86}$$

As can be seen, there is a summation over the indices μ and the ∂_μ can be considered as a basis of the tangent space. This is sometimes applied in mathematical textbooks (for example [10]).

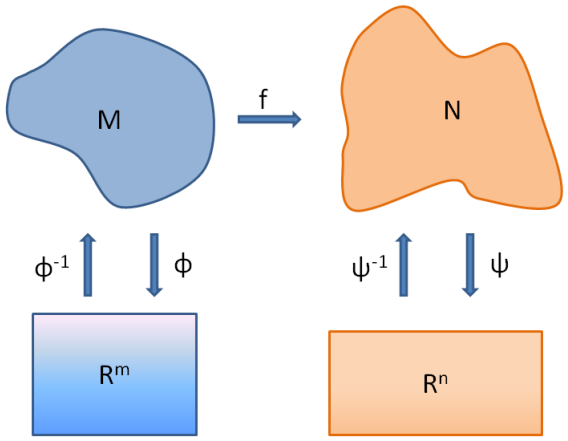

Figure 2.6: Mapping between two manifolds and tangent spaces.

n-forms

There is a special class of tensors, called n-forms. These comprise all completely anti-symmetric covariant tensors. In an n-dimensional space, there are 0-forms, 1-forms, ..., n-forms. All higher forms are zero by the antisymmetry requirement. A 2-form F can be constructed, for example, by two 1-forms (co-vectors) a and b:

$$F_{\mu\nu} = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu). \tag{2.87}$$

By index raising this can be rewritten to the form

$$F'^{\mu\nu} = \frac{1}{2}(a'^\mu b'^\nu - a'^\nu b'^\mu) \tag{2.88}$$

with

$$a'^\mu = g^{\mu\nu}a_\nu, \text{ etc.} \tag{2.89}$$

Introducing a square bracket for an antisymmetric index permutation:

$$[\mu\nu] \to \mu\nu - \nu\mu \tag{2.90}$$

we can also write this in the form

$$F_{\mu\nu} = \frac{1}{2}a_{[\mu}b_{\nu]}. \tag{2.91}$$

In general, we can define

$$T_{[\mu_1\mu_2\ldots\mu_n]} = \frac{1}{n!}(T_{\mu_1\mu_2\ldots\mu_n} + \text{alternating sum over permutations of } \mu_1\ldots\mu_n). \tag{2.92}$$

The antisymmtric tensor may contain further indices which are not permuted.

■ **Example 2.8** Consider a tensor $T_{\mu\nu\rho\sigma}{}^{\tau}$ being antisymmetric in the first three indices. Then we have

$$T_{[\mu\nu\rho]\sigma}{}^{\tau} = \frac{1}{6}(T_{\mu\nu\rho\sigma}{}^{\tau} - T_{\mu\rho\nu\sigma}{}^{\tau} + T_{\rho\mu\nu\sigma}{}^{\tau} - T_{\nu\mu\rho\sigma}{}^{\tau} + T_{\nu\rho\mu\sigma}{}^{\tau} - T_{\rho\nu\mu\sigma}{}^{\tau}). \tag{2.93}$$

By utilizing the antisymmetry of the first two indices, we can simplify this expression to

$$\begin{aligned} T_{[\mu\nu\rho]\sigma}{}^{\tau} =& \frac{1}{6}\left(T_{\mu\nu\rho\sigma}{}^{\tau} - (-T_{\mu\nu\rho\sigma}{}^{\tau}) + T_{\rho\mu\nu\sigma}{}^{\tau} - (-T_{\rho\mu\nu\sigma}{}^{\tau}) + T_{\nu\rho\mu\sigma}{}^{\tau} - (-T_{\nu\rho\mu\sigma}{}^{\tau})\right) \\ =& \frac{1}{3}\left(T_{\mu\nu\rho\sigma}{}^{\tau} + T_{\rho\mu\nu\sigma}{}^{\tau} + T_{\nu\rho\mu\sigma}{}^{\tau}\right). \end{aligned} \tag{2.94}$$

This is the sum of indices μ,ν,ρ cyclically permuted. ■

With the help of antisymmetrization, we can define the *exterior product* or *wedge product*. Given a p-form a and q-form b, we define the antisymmetric product by the $\wedge$ (*wedge*) operator:

$$(a\wedge b)_{\mu_1\ldots\mu_{p+q}} := \frac{(p+q)!}{p!q!}a_{[\mu_1\ldots\mu_p}b_{\mu_{p+1}\ldots\mu_{p+q}]}. \tag{2.95}$$

For example, the wedge product of two 1-forms is

$$(a\wedge b)_{\mu\nu} = 2a_{[\mu}b_{\nu]} = a_\mu b_\nu - a_\nu b_\mu. \tag{2.96}$$

The wedge product is associative:

$$(a\wedge(b+c))_{\mu\nu} = (a\wedge b)_{\mu\nu} + (a\wedge c)_{\mu\nu}. \tag{2.97}$$

Mathematicians like to omit the indices if it is clear that an equation is written for forms. Thus the last equation can also be written as

$$a\wedge(b+c) = a\wedge b + a\wedge c \tag{2.98}$$

in a short-hand notation. Another property is that wedge products are not commutative. For a p-form a and a q-form b it is

$$a\wedge b = (-1)^{pq}b\wedge a \tag{2.99}$$

and for a 1-form:

$$a\wedge a = 0. \tag{2.100}$$

These features may justify the name "exterior product" as a generalization of a vector product in three dimensions.

An important operation on forms is applying the *Hodge dual*. First we have to define the Levi-Civita symbol in n dimensions:

$$\varepsilon_{\mu_1\ldots\mu_n} = \begin{cases} 1 & \text{if } \mu_1\ldots\mu_n \text{ is an even permutation of } 0,\ldots(n-1), \\ -1 & \text{if } \mu_1\ldots\mu_n \text{ is an odd permutation of } 0,\ldots(n-1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.101}$$

The determinant of a matrix can be expressed by this symbol. If $M^{\mu}_{\mu'}$ is a $n \times n$ matrix, the determinant $|M|$ obeys the relation

$$\varepsilon_{\mu'_1 \dots \mu'_n} |M| = \varepsilon_{\mu_1 \dots \mu_n} M^{\mu_1}_{\mu'_1} \cdots M^{\mu_n}_{\mu'_n} \tag{2.102}$$

or, restricting to one permutation at the left-hand side:

$$|M| = \varepsilon_{\mu_1 \dots \mu_n} M^{\mu_1}_{1} \cdots M^{\mu_n}_{n}. \tag{2.103}$$

The Levi-Civita symbol is defined in any coordinate system in the same way, not undergoing a coordinate transformation. Therefore, it is not a tensor. The symbol is totally antisymmetric, i.e. when any two indices are interchanged, the sign changes. All elements where one index appears twice are zero because the index set must be a permutation.

The Levi-Civita symbol can also be defined with upper indices in the same way. Then the determinant (2.102/2.103) takes the form

$$\varepsilon^{\mu'_1 \dots \mu'_n} |M| = \varepsilon^{\mu_1 \dots \mu_n} M^{\mu'_1}_{\mu_1} \cdots M^{\mu'_n}_{\mu_n} \tag{2.104}$$

or

$$|M| = \varepsilon^{\mu_1 \dots \mu_n} M^{1}_{\mu_1} \cdots M^{n}_{\mu_n}. \tag{2.105}$$

We can construct a tensor from the Levi-Civita symbol by multiplying it with the square root of the modulus of the metric (in Minkowski space the metric is negative definite, therefore we have to take the modulus). To show this, we start with the transformation equation of the metric tensor

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} \tag{2.106}$$

and apply the determinant. With the product rule of determinants this can be written as

$$|g_{\mu'\nu'}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| \left| \frac{\partial x^{\nu}}{\partial x^{\nu'}} \right| |g_{\mu\nu}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|^2 |g_{\mu\nu}| \tag{2.107}$$

or

$$\left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \sqrt{\frac{|g_{\mu'\nu'}|}{|g_{\mu\nu}|}} \tag{2.108}$$

where the left-hand side represents the determinant of the Jacobian. Using the special case

$$M^{\mu'}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \tag{2.109}$$

and inserting this into (2.104) we obtain

$$\varepsilon^{\mu'_1 \dots \mu'_n} \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| = \varepsilon^{\mu_1 \dots \mu_n} \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}. \tag{2.110}$$

The determinant of the inverse Jacobian is

$$\left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-1}, \tag{2.111}$$

therefore we obtain from (2.110) with inserting (2.108):

$$\varepsilon^{\mu'_1\dots\mu'_n}\frac{1}{\sqrt{|g_{\mu'\nu'}|}}=\varepsilon^{\mu_1\dots\mu_n}\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}\frac{1}{\sqrt{|g_{\mu\nu}|}}. \tag{2.112}$$

So $\varepsilon^{\mu_1\dots\mu_n}/\sqrt{|g|}$ transforms like a tensor, and therefore is a tensor, by definition. The corresponding covariant tensor transforms as

$$\varepsilon_{\mu'_1\dots\mu'_n}\sqrt{|g_{\mu'\nu'}|}=\varepsilon_{\mu_1\dots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}\sqrt{|g_{\mu\nu}|}. \tag{2.113}$$

Indices can be raised and lowered as usual by multiplying with metric elements.

With this behavior of the Levi-Civita symbol in mind, we define the Hodge-Dual of a tensorial form as follows. Assume a n-dimensional manifold, a p-dimensional sub-manifold $p<n$, and a tensor p-form A. We then define

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}}:=\frac{1}{p!}|g|^{-1/2}\,\varepsilon^{\nu_1\dots\nu_p}{}_{\mu_1\dots\mu_{n-p}}A_{\nu_1\dots\nu_p}. \tag{2.114}$$

The tilde superscript $\widetilde{\ }$ is called the *Hodge dual operator*. In the mathematical literature this is mostly denoted by an asterisk as prefix-operator ($*A$) but this is a very misleading notation, therefore we prefer the tilde superscript. The Hodge dual can be rewritten with a Levi-Civita symbol with only covariant components by

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}}=\frac{1}{p!}|g|^{-1/2}\,g^{\nu_1\sigma_1}\cdots g^{\nu_p\sigma_p}\,\varepsilon_{\sigma_1\dots\sigma_p\mu_1\dots\mu_{n-p}}A_{\nu_1\dots\nu_p}. \tag{2.115}$$

In this book, we will mostly use a somewhat simpler form where a contravariant tensor is transformed into a covariant tensor and vice versa. The factors $g^{\nu_1\sigma_1}$, etc., can be used to raise the indices of $A_{\nu_1\dots\nu_p}$:

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}}=\frac{1}{p!}|g|^{-1/2}\,\varepsilon_{\nu_1\dots\nu_n}A^{\nu_1\dots\nu_p}, \tag{2.116}$$

$$\widetilde{A}^{\mu_1\dots\mu_{n-p}}=\frac{1}{p!}|g|^{1/2}\,\varepsilon^{\nu_1\dots\nu_n}A_{\nu_1\dots\nu_p} \tag{2.117}$$

where the sign of the exponent of $|g|$ has been changed according to (2.113). As an example, in four-dimensional space we use $n=4, p=2$. Then Hodge duals of the A form are

$$\widetilde{A}_{\mu\nu}=\frac{1}{2}|g|^{-1/2}\,\varepsilon_{\mu\nu\sigma\rho}A^{\sigma\rho}, \tag{2.118}$$

$$\widetilde{A}^{\mu\nu}=\frac{1}{2}|g|^{1/2}\,\varepsilon^{\mu\nu\sigma\rho}A_{\sigma\rho}. \tag{2.119}$$

The Hodge dual $\widetilde{A}$ is linearly independent on the original form A. We will use the Hodge dual when deriving the theorems of Cartan geometry and the field equations of ECE theory.

2.4 Differentiation

We have already used some types of differentiation in the preceding sections, but only in the "standard" way within Euclidean spaces. Now we will extend this to curved spaces (manifolds) and to the calculus of p-forms.

2.4.1 Covariant differentiation

So far, we have already used partial derivatives of tensors and parametrized derivatives. This, however, is not sufficient to define a general type of derivative in curved spaces of manifolds. Partial derivatives depend on the coordinate system. What we need is a "generally covariant" derivative that keeps its form under coordinate transformations and passes into the partial derivative for Euclidean spaces.

To retain linearity, the covariant derivative should have the form of a partial derivative plus a linear transformation. The latter corrects the partial derivative in such a way that covariance is ensured. The linear transformation depends on the coordinate indices. We define for the *covariant derivative* of an arbitrary vector field V^ν:

$$D_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{2.120}$$

where the $\Gamma^\nu_{\mu\lambda}$ are functions and called the *connection coefficients* or *Christoffel symbols*. In contrast to an ordinary partial derivative, the covariant derivative of a vector component V^ν depends on all other components via the sum with the connection coefficients (observe the summation convention!). The covariant derivative has tensor properties by definition, therefore Eq. (2.120) is a tensor equation, transducing a (1,0) tensor into a (1,1) tensor, and we can apply the transformation rules for tensors:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} D_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\frac{\partial}{\partial x^\mu} V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \right). \tag{2.121}$$

On the other hand, we can apply the transformation to Eq. (2.120) directly:

$$D_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}. \tag{2.122}$$

The single terms on the right-hand side transform as follows:

$$\begin{aligned} \partial_{\mu'} V^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial}{\partial x^\mu} V^\nu, \end{aligned} \tag{2.123}$$

$$\Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} = \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda, \tag{2.124}$$

where the product rule has been applied in the first term. Eqs. (2.122) and (2.121) can be equated. The term with the partial derivative of V^ν cancels out and we obtain:

$$\Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{2.125}$$

Here we have replaced the dummy index ν by λ in the term with the mixed partial derivative. This is a common operation for tensor equations. Another common operation is multiplying a tensor equation by an indexed term and summing over one or more free indices (i.e. making the previously independent index into a dummy index). Multiplying the last equation by $\frac{\partial x^\lambda}{\partial x^{\lambda'}}$ then gives

$$\Gamma^{\nu'}_{\mu'\lambda'} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda \tag{2.126}$$

so that we approach an equation of determining the transformation of the connection coefficients. The last equation holds for any vector V^λ, therefore the equation must hold for the coefficients of V^λ directly. Thus, we obtain the transformation equation for the connection coefficients:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{2.127}$$

Obviously, the Gammas do not transform as a tensor, the last term prevents this. The Gammas are not a tensor, therefore indices of Gamma cannot be raised and lowered by multiplying with metric elements and we need not put too much effort into maintaining the order of upper and lower indices.

So far, we have investigated covariant derivatives of a contravariant vector (Eq. (2.120)). The theory can be extended to covariant vectors of 1-forms ω_ν:

$$D_\mu \omega_\nu := \partial_\mu \omega_\nu + \bar{\Gamma}^\lambda_{\mu\nu} \omega_\lambda \tag{2.128}$$

where $\bar{\Gamma}$ is a connection coefficient being a priori different from Γ. It can be shown [5] that, for consistency reasons, $\bar{\Gamma}$ is the same as Γ except for the sign:

$$\bar{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu}. \tag{2.129}$$

Please note that the summation indices are different between (2.120) and (2.128). Now that we have a covariant derivative for contravariant and covariant components, the covariant derivative for arbitrary (k,m) tensors is defined as follows:

$$\begin{aligned} D_\sigma T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} := \partial_\sigma T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \mu_3 \ldots \mu_k}{}_{\nu_1 \ldots \nu_m} + \ldots \\ - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \ldots \mu_k}{}_{\lambda \nu_2 \ldots \nu_m} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \lambda \nu_3 \ldots \nu_m} - \ldots \end{aligned} \tag{2.130}$$

By applying the covariant derivative, a (k,m) tensor is transformed into a (k,m+1) tensor. It is also possible to take the covariant derivative of a scalar function. Since no indices are defined for the connection in this case, we define for a scalar function ϕ:

$$D_\mu \phi := \partial_\mu \phi. \tag{2.131}$$

As we have seen, the connection coefficients are not a tensor. It is, however, easy to make a tensor of them by taking the antisymmetric sum of the lower indices:

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{2.132}$$

This is called the *torsion tensor*. When applying the transformation (2.127) for the difference of Gammas, the last term vanishes because the order in the mixed partial derivative is arbitrary. The torsion tensor is antisymmetric by definition. In four dimensions it can be written out for each index λ as

$$(T^\lambda{}_{\mu\nu}) = \begin{bmatrix} 0 & T^\lambda{}_{01} & T^\lambda{}_{02} & T^\lambda{}_{03} \\ -T^\lambda{}_{01} & 0 & T^\lambda{}_{12} & T^\lambda{}_{13} \\ -T^\lambda{}_{02} & -T^\lambda{}_{12} & 0 & T^\lambda{}_{23} \\ -T^\lambda{}_{03} & -T^\lambda{}_{13} & -T^\lambda{}_{23} & 0 \end{bmatrix}. \tag{2.133}$$

There are six independent components per λ. We will see later that this is one of the basis elements of Cartan geometry. A connection that is symmetric in its lower indices is torsion-free.

For completeness, we give the definition of the *Riemann curvature tensor*, which is also defined by the connection coefficients, but in a more complicated manner:

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho}. \tag{2.134}$$

The tensor is antisymmetric in its last two indices. If it is written in pure covariant form $R_{\lambda\rho\mu\nu} = g_{\tau\lambda} R^\tau{}_{\rho\mu\nu}$ and the manifold is torsion-free, the Riemann tensor is also antisymmetric in its first two indices. This property will, however, not be used in Cartan geometry.

2.4.2 Metric compatibility and parallel transport

A fundamental property of vectors in physics is that they must be independent of their coordinate representation. From Euclidean space, we know that a rotation of a vector leaves its length and orientation against other vectors constant. In curved manifolds this is not necessarily the case anymore. Whether the length of a vector is preserved depends on the metric tensor. A parallel transport of a vector is depicted in Fig. 2.7. On a spherical surface, a vector is parallel transported from the north pole to a point on the equator in two ways: 1) moved directly along a meridian (red; right) and 2) moved first along another meridian and then along an equatorial latitude (left; blue). Obviously, the results are different, so this naive procedure is not compatible with a spherical manifold.

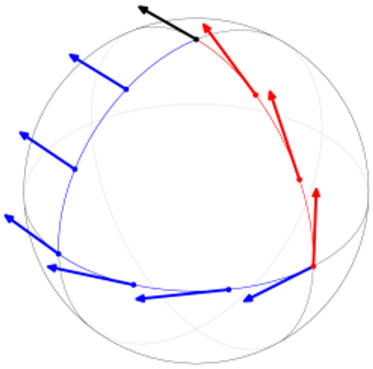

Figure 2.7: Parallel transport of a vector on a sphere.

Let us formalize the process to define parallel transport in a compatible way. A path is a displacement of a vector V^ν whose coordinates are parameterized, say by a parameter λ:

$$V^\nu = V^\nu(\lambda) \text{ at point } x^\nu(\lambda). \tag{2.135}$$

This can be considered as moving the vector (which is a tensor) along a predefined path. We define the covariant derivative along the path by

$$\frac{D}{d\lambda} := \frac{dx^\mu}{d\lambda} D_\mu, \tag{2.136}$$

where $\frac{dx^\mu}{d\lambda}$ is the tangent vector of the path. This gives us a method for specifying a *parallel transport* of V. This transport condition is fulfilled if the covariant derivative along the path vanishes:

$$\frac{D\,V^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu V^\nu = \frac{dx^\mu}{d\lambda}(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) = 0. \tag{2.137}$$

Since the tangent vector cannot vanish (we would not have a path anymore), it follows that the covariant derivative of the tensor must vanish:

$$D_\mu V^\nu = 0. \tag{2.138}$$

This is the condition for parallel transport. It is fulfilled if and only if the covariant derivative along a path vanishes. This holds for any tensor. In particular we can choose the metric tensor and require it to be parallel transported:

$$D_\sigma g_{\mu\nu} = 0. \tag{2.139}$$

This is called *metric compatibility*. It is also said that the connection is metrically compatible because it is contained in the covariant derivative. It means that the metric tensor is covariantly constant everywhere and can be parallel transported. If this requirement were omitted, we would have difficulties defining meaningful physics in a manifold, for example, norms of vectors would not be constant but change during translations or rotations.

■ **Example 2.9** We show that the inner product of two vectors is preserved if the vectors can be parallel transported. The inner product of vectors V^μ and W^ν is $g_{\mu\nu}V^\mu W^\nu$. Its covariant path derivative is

$$\frac{D}{d\lambda}\left(g_{\mu\nu}V^\mu W^\nu\right) = \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right) = 0 \tag{2.140}$$

because all three tensors are parallel transported, by definition. In the same way, one can prove that, if $g_{\mu\nu}$ can be parallel transported, then so can its inverse $g^{\mu\nu}$:

$$\begin{aligned} 0 = \frac{D}{d\lambda}g_{\mu\nu} &= \frac{D}{d\lambda}\left(g_{\mu\sigma}\, g_{\rho\nu}\, g^{\rho\sigma}\right) \\ &= \frac{D}{d\lambda}\left(g_{\mu\sigma}\right)g_{\rho\nu}\, g^{\rho\sigma} + g_{\mu\sigma}\frac{D}{d\lambda}\left(g_{\rho\nu}\right)g^{\rho\sigma} + g_{\mu\sigma}\, g_{\rho\nu}\frac{D}{d\lambda}\left(g^{\rho\sigma}\right). \end{aligned} \tag{2.141}$$

The first two terms in the last line vanish by definition, and consequently the third term has to vanish. ■

The concept of parallel transport allows us to find the equation for geodesics. A *geodesic* is the generalization of a straight line in Euclidean space. Mass points without external forces move this way. In a curved manifold, the motion follows the curving of space and therefore is not a straight line. We can find the equation of geodesics by requiring that the path parallel transports its own tangent vector. This is in analogy to flat space where the tangent vector is parallel to its line vector. From (2.137) then we have

$$\frac{D}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.142}$$

which can be written

$$\frac{dx^\mu}{d\lambda}D_\mu\frac{dx^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\left(\frac{\partial}{\partial x^\mu}\frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\rho}\frac{dx^\rho}{d\lambda}\right) = 0 \tag{2.143}$$

and, by replacement of $\frac{\partial}{\partial x^\mu}$ by $\frac{\partial\lambda}{\partial x^\mu}\frac{d}{d\lambda}$, simplifies to

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \tag{2.144}$$

which is the *geodesic equation*. In flat space, the Gammas vanish and Newton's law $\ddot{x} = 0$ for an unconstrained motion is regained.

Given a path in the manifold, covariant derivatives can be used to describe the deviation of a tensor from being parallel transported. Consider a round-trip as depicted in Fig. 2.8. A tensor is moved counter-clockwise along its covariant tangent vector D_μ, then D_ν, and afterwards back to its starting point in the reverse order. In the case where the tensor is parallel transportable, all derivatives vanish. However, this will not be the case in general. The *commutator* of two covariant derivatives is defined as

$$[D_\mu, D_\nu] := D_\mu D_\nu - D_\nu D_\mu \tag{2.145}$$

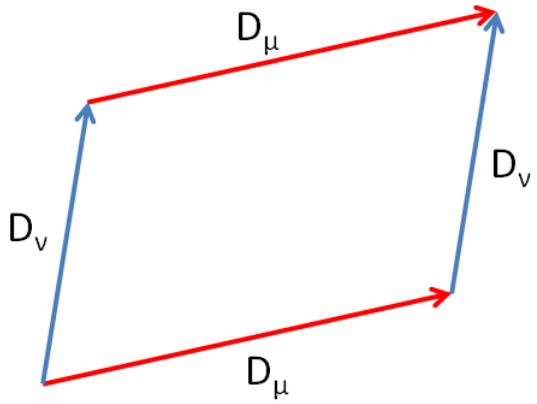

Figure 2.8: Closed loop for composition of two covariant derivatives.

and describes the difference of both paths with respect to the covariant derivative. We can apply this to a vector V^ρ and evaluate the terms:

$$\begin{aligned}
[D_\mu, D_\nu]V^\rho &= D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho \\
&= \partial_\mu (D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma \\
&\quad - \partial_\nu (D_\mu V^\rho) + \Gamma^\lambda_{\nu\mu} D_\lambda V^\rho - \Gamma^\rho_{\nu\sigma} D_\mu V^\sigma \\
&= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda \\
&\quad - \partial_\nu \partial_\mu V^\rho - (\partial_\nu \Gamma^\rho_{\mu\sigma}) V^\sigma - \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho + \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda \\
&= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) V^\sigma - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) D_\lambda V^\rho .
\end{aligned} \tag{2.146}$$

Comparing the last line with the definitions of curvature tensor (2.134) and torsion tensor (2.132), it can be written:

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho . \tag{2.147}$$

Interestingly, the commutator of covariant derivatives of a vector depends linearly on the vector itself and its tangent vector, where the coefficients are the curvature and torsion tensors. In the case of no torsion, there would be no dependence on a derivative of V^ρ at all. The action of $[D_\mu, D_\nu]$ can be applied to a tensor of arbitrary rank. In general, it is

$$\begin{aligned}
[D_\rho, D_\sigma] X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_m} &= R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_m} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1 \lambda \dots \mu_k}{}_{\nu_1 \dots \nu_m} + \cdots \\
&\quad - R^\lambda{}_{\nu_1\rho\sigma} X^{\mu_1 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_m} - R^\lambda{}_{\nu_2\rho\sigma} X^{\mu_1 \dots \mu_k}{}_{\nu_1 \lambda \dots \nu_m} - \cdots \\
&\quad - T^\lambda{}_{\rho\sigma} D_\lambda X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_m} .
\end{aligned} \tag{2.148}$$

We have seen that the curvature and torsion tensors depend on the connection coefficients directly. To describe the geometry of a manifold, one must know these coefficients. The geometry is typically defined by a coordinate transformation. However, there is no direct way to derive the connection coefficients from the coordinate transformation equations. In Example (2.8) we have seen that the metric tensor can be derived from the Jacobian, which contains the derivatives of the coordinate transformations. Therefore, what we need is a relation between the metric and the

connection, from which the connection coefficients can be derived, when the metric is known. Such a relation is given by the metric compatibility condition:

$$D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} g_{\mu\lambda} = 0. \tag{2.149}$$

For a space of four dimensions, this tensor equation represents $4^3 = 64$ single equations. The first of them (for the diagonal metric elements) read:

$$\begin{aligned} &\frac{\partial}{\partial x^0} g_{00} - 2\Gamma^0_{00}\, g_{00} = 0 \\ &-\Gamma^1_{00}\, g_{11} - \Gamma^0_{01}\, g_{00} = 0 \\ &-\Gamma^2_{00}\, g_{22} - \Gamma^0_{02}\, g_{00} = 0 \\ &-\Gamma^3_{00}\, g_{33} - \Gamma^0_{03}\, g_{00} = 0 \\ &\qquad\qquad \dots \end{aligned} \tag{2.150}$$

You should keep in mind that the metric is symmetric, and therefore not all equations are linearly independent. It is difficult to see how many independent equations remain. Computer algebra (code available at [8]) tells us that one half (24 equations) are dependent on the other 24 equations. Therefore, we can predefine 24 Gammas arbitrarily. A solution is, for example,

$$\begin{aligned} \Gamma^0_{00} &= \frac{\frac{\partial}{\partial x^0}\, g_{00}}{2\, g_{00}} \\ \Gamma^0_{01} &= -\frac{g_{11}}{g_{00}} A_{25} \\ \Gamma^0_{02} &= -\frac{g_{22}}{g_{00}} A_{43} \\ \Gamma^0_{03} &= -\frac{g_{33}}{g_{00}} A_{40} \\ \Gamma^0_{10} &= \frac{\frac{\partial}{\partial x^1}\, g_{00}}{2\, g_{00}} \\ &\dots \end{aligned} \tag{2.151}$$

with

$$\begin{aligned} \Gamma^1_{00} &= A_{25} \\ \Gamma^2_{00} &= A_{43} \\ \Gamma^3_{00} &= A_{40} \\ &\dots \end{aligned} \tag{2.152}$$

where the A_i are the predefined parameters, and they may even be functions of x^μ. From the first equation of (2.150), it can be seen that assuming $\Gamma^0_{00} = 0$ is not a good choice, because this would impose the restriction $\frac{\partial g_{00}}{\partial x^0} = 0$ on the metric a priori. Therefore, the diagonal elements of the lower pair of indices of Gamma do not vanish in general. By comparing the solutions for Γ^0_{01} andΓ^0_{10} in (2.151) it is obvious that the Gammas are not symmetric in the lower indices.

Having found the connection coefficients, we can construct the curvature and torsion tensors (2.134) and (2.132). While the coefficients go into the curvature tensor as is, the torsion tensor itself depends only on the antisymmetric part of the Gammas. Each 2-tensor or connection coefficient can be split into a symmetric and antisymmetric part:

$$\Gamma^\rho_{\mu\nu} = \Gamma^{\rho(S)}_{\mu\nu} + \Gamma^{\rho(A)}_{\mu\nu} \tag{2.153}$$

with

$$\Gamma^{\rho(S)}_{\mu\nu} = \Gamma^{\rho(S)}_{\nu\mu}, \qquad \Gamma^{\rho(A)}_{\mu\nu} = -\Gamma^{\rho(A)}_{\nu\mu}. \tag{2.154}$$

For the torsion tensor we have

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda(A)}_{\mu\nu} - \Gamma^{\lambda(A)}_{\nu\mu} = 2\,\Gamma^{\lambda(A)}_{\mu\nu}, \tag{2.155}$$

the symmetric part does not enter the torsion. This motivates the imposition of additional antisymmetry requirements on the Gammas, instead of choosing 24 elements arbitrarily. So, in addition to the metric compatibility equation (2.149), we define 24 extra equations

$$\Gamma^{\rho}_{\mu\nu} = -\Gamma^{\rho}_{\nu\mu} \tag{2.156}$$

for all pairs $\mu \neq \nu$ with $\mu > \nu$. This reduces the number of free solution parameters from 24 to 4 (see computer algebra code [49]). The situation gets quite complicated when non-diagonal elements in the metric are present [50]. Alternatively, we could even force a purely symmetric connection by requiring that

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}. \tag{2.157}$$

Then there are no free parameters anymore, and all Gammas are uniquely defined, where 24 of them turn out to be zero. However, in this case, torsion is zero and we will run into irretrievable conflicts with geometrical laws, as we will see in subsequent sections. There is a reason for leaving a certain variability in the connection: the theorems of Cartan geometry have to be satisfied, which imposes additional conditions on curvature and torsion, and thereby on the connection.

For completeness, we describe how the symmetric connection coefficients are computed in Einsteinian general relativity. Starting with Eq. (2.149), this equation is written three times with permuted indices:

$$\begin{aligned} &\partial_\sigma g_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu} g_{\mu\lambda} = 0, \\ &\partial_\mu g_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\nu\lambda} = 0, \\ &\partial_\nu g_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\sigma\lambda} = 0. \end{aligned} \tag{2.158}$$

Subtracting the second and third equation from the first and using the symmetry of the connection gives

$$\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu} + 2\,\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} = 0, \tag{2.159}$$

and multiplying the equation by $g^{\sigma\rho}$ gives for the Gamma term:

$$(\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma})\, g^{\sigma\rho} = \Gamma^{\lambda}_{\mu\nu} (g_{\lambda\sigma}\, g^{\sigma\rho}) = \Gamma^{\lambda}_{\mu\nu} \delta^{\rho}_{\lambda} = \Gamma^{\rho}_{\mu\nu}. \tag{2.160}$$

From (2.159) then follows

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{2.161}$$

The symmetric connection is determined completely by the metric, in accordance with our earlier result from the single equation of metric compatibility.

All derivations in this section were exemplified with a diagonal metric. They remain true if non-diagonal elements are added, but the solutions become much more complex. Imposing

additional symmetry or antisymmetry conditions on the connection may lead to results differing from those for a diagonal metric.

It should be noted that the metric of a given geometry of a manifold is not unique, and depends on the choice of coordinate system. Recalling the examples above, Euclidean space can be described by cartesian or spherical coordinates which lead to different metric tensors. However, the spacetime structure is the same, only the numerical addressing of points changes, as do the coordinates of vectors. However, the vectors as physical objects (position and length) remain the same.

■ **Example 2.10** We compute the connection for the spherical coordinate system (r,θ,ϕ) for three cases: general connection, antisymmetrized connection, and symmetrized connection. This example is available as Maxima code [48]. The metric tensor is from Example 2.5:

$$(g_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix}. \tag{2.162}$$

Since the metric is not time-dependent, indices run from 1 to 3. This gives $3^3 = 27$ equations from metric compatibility (2.149), and the first equations are

$$\begin{aligned} -2\,\Gamma^1_{11} &= 0 \\ -\Gamma^2_{11}\,r^2 - \Gamma^1_{12} &= 0 \\ -\Gamma^3_{11}\,r^2\sin^2(\theta) - \Gamma^1_{13} &= 0 \\ -\Gamma^2_{11}\,r^2 - \Gamma^1_{12} &= 0 \\ 2r - 2\,\Gamma^2_{12}\,r^2 &= 0 \\ &\dots \end{aligned} \tag{2.163}$$

The solution (obtained by computer algebra) contains 9 free parameters $A_1,\dots,A_9$. There are 27 solutions in total. Some of them are:

$$\begin{aligned} \Gamma^1_{11} &= 0 \\ \Gamma^1_{13} &= -A_9\,r^2\sin^2(\theta) \\ \Gamma^1_{31} &= 0 \\ \Gamma^1_{33} &= -A_3\,r^2\sin^2(\theta) \\ \Gamma^2_{12} &= \frac{1}{r} \\ \Gamma^2_{21} &= -\frac{A_4}{r^2} \\ \Gamma^3_{23} &= \frac{\cos(\theta)}{\sin(\theta)} \\ \Gamma^3_{32} &= A_2 \\ &\dots \end{aligned} \tag{2.164}$$

With 9 additional antisymmetry conditions, the solutions are

$$\begin{aligned}
&\Gamma^1_{11} = 0 \\
&\Gamma^1_{13} = \Gamma^1_{31} = 0 \\
&\Gamma^1_{23} = -\Gamma^1_{32} = -A_{10} \\
&\Gamma^2_{12} = -\Gamma^2_{21} = \frac{1}{r} \\
&\Gamma^3_{23} = -\Gamma^3_{32} = \frac{\cos(\theta)}{\sin(\theta)} \\
&\dots
\end{aligned} \tag{2.165}$$

There is only one free parameter A_{10} left. A certain similarity to the general solution is retained, but with antisymmetry. If symmetric connection coefficients are enforced, most Gammas are zero. The only non-zero coefficients are:

$$\begin{aligned}
&\Gamma^1_{22} = -r \\
&\Gamma^1_{33} = -r\sin^2(\theta) \\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \\
&\Gamma^2_{33} = -\cos(\theta)\sin(\theta) \\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos(\theta)}{\sin(\theta)}
\end{aligned} \tag{2.166}$$

This example is often found in textbooks of general relativity. If all coordinates have the physical dimension of length, then the connection coefficients have the same physical dimension. In this example we have angles and lengths, therefore the physical dimensions differ. ∎

2.4.3 Exterior derivative

So far, we have dealt with covariant derivatives of tensors. Now we want to extend the concept of derivatives to n-forms. We already know that a partial derivative of a tensor does not conserve the tensor properties. Therefore, we will define an appropriate derivative for n-forms. We have already introduced antisymmetric forms in Section 2.3. It is useful to define a derivative on these objects that conserves antisymmetry and tensor properties. A partial derivative for one coordinate generates an additional index in a tensor, therefore a p-form is extended to a (p+1)-form by the definition

$$(d \wedge A)_{\mu_1 \dots \mu_{p+1}} := (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}. \tag{2.167}$$

This (p+1)-form is a tensor, irrespective of what A is. The simplest exterior derivative is that of a scalar function $\phi(x_\mu)$ which is

$$(d \wedge \phi)_\mu = \partial_\mu \phi, \tag{2.168}$$

in other words, this is the gradient of ϕ. Another example is the definition of the electromagnetic field in tensor form $F_{\mu\nu}$ as a 2-form (see Example 2.11 below). It is derived as an exterior derivative of a 1-form, the vector potential A_μ:

$$F_{\mu\nu} := (d \wedge A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.169}$$

The tensor character of exterior derivatives can be seen by applying the transformation law (2.123) to a (0,1) tensor V for example:

$$\begin{aligned}\frac{\partial}{\partial x^{\mu'}}V_{\nu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}V_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}V_{\nu}\right) \\ &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu}}{\partial x^{\mu}\partial x^{\nu'}}V_{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\mu}}V_{\nu}.\end{aligned} \tag{2.170}$$

The first term in the second line should not appear if this were a tensor transformation. It can be rewritten to

$$\frac{\partial^2 x^{\nu}}{\partial x^{\mu'}\partial x^{\nu'}}V_{\nu} \tag{2.171}$$

and now is symmetric in μ' and ν'. Since the exterior derivative only contains antisymmetric sums of both indices, all these terms vanish because partial derivatives are commutable. Therefore, $d\wedge V_{\nu}$ transforms like a tensor, and so do all n-forms.

An important property of an exterior derivative is that its two-fold application is zero:

$$d\wedge(d\wedge A) = 0. \tag{2.172}$$

The reason is the same as above, the partial derivatives are commutable, summing up to zero in all antisymmetric sums.

■ **Example 2.11** We describe Maxwell's homogeneous field equations in form notation and transform this to the well-known vector form (see computer algebra code [51]). The homogeneous laws are the Gauss law and the Faraday law. In tensor notation they are condensed into one equation:

$$d\wedge F = 0 \tag{2.173}$$

or with indices

$$(d\wedge F)_{\mu\nu\rho} = 0. \tag{2.174}$$

Because F is a 2-form, the exterior derivative of F is a 3-form. The electromagnetic field tensor is antisymmetric and defined by the contravariant tensor

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \tag{2.175}$$

where E^i are the components of the electric field and B^i those of the magnetic field. It is $E^1 = E_X, E^2 = E_Y$, etc. To be able to apply the exterior derivative, we first have to transform this tensor to covariant form. Since classical electrodynamics takes place in a Euclidean space, we use the Minkowski metric to lower the indices:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.176}$$

Then the covariant field tensor is

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -cB^3 & cB^2 \\ -E^2 & cB^3 & 0 & -cB^1 \\ -E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}. \tag{2.177}$$

Compared to the contravariant form, only the signs of the electric field components have changed. Working out the exterior derivative for $\mu = 0, \nu = 1, \rho = 2$, we obtain

$$(d \wedge F)_{012} = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} - \partial_0 F_{21} - \partial_1 F_{02} - \partial_2 F_{10}. \tag{2.178}$$

Because F is antisymmetric, the negative summands are equal to the positive summands with reversed sign so that we have

$$(d \wedge F)_{012} = 2(\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01}), \tag{2.179}$$

this is twice the cyclic sum of indices. Since (μ, ν, ρ) must be a subset of (0, 1, 2, 3) only the combinations
(0, 1, 2)
(0, 1, 3)
(0, 2, 3)
(1, 2, 3)
are possible, leading to four equations for $d \wedge F$. Setting $F_{01} = E_X$ etc. leads to the four equations:

$$\begin{aligned} 2(c\partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1) &= 0 \\ 2(-c\partial_0 B^2 + \partial_1 E^3 - \partial_3 E^1) &= 0 \\ 2(c\partial_0 B^1 + \partial_2 E^3 - \partial_3 E^2) &= 0 \\ 2(c\partial_1 B^1 + c\partial_2 B^2 + c\partial_3 B_3) &= 0 \end{aligned} \tag{2.180}$$

or, written with cartesian components and simplified:

$$\begin{aligned} \partial_t B_Z + \partial_X E_Y - \partial_Y E_X &= 0 \\ \partial_t B_Y - \partial_X E_Z + \partial_Z E_X &= 0 \\ \partial_t B_X + \partial_Y E_Z - \partial_Z E_Y &= 0 \\ \partial_X B_X + \partial_Y B_Y + \partial_Z B_Z &= 0 \end{aligned} \tag{2.181}$$

where we have used $\partial_0 = 1/c \cdot \partial_t$. Comparing these equations with the curl operator:

$$\nabla \times \mathbf{V} = \begin{bmatrix} \partial_Y V_Z - \partial_Z V_Y \\ -\partial_X V_Z + \partial_Z V_X \\ \partial_X V_Y - \partial_Y V_X \end{bmatrix} \tag{2.182}$$

the first three equations of (2.181) contain the third, second and first line of this operator and can be written in vector form:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{2.183}$$

which is the Faraday law. The fourth equation of (2.181) is the Gauss law

$$\nabla \cdot \mathbf{B} = 0. \tag{2.184}$$

We conclude this example with the hint that the inhomogeneous Maxwell equations (Coulomb law and Ampère-Maxwell law) cannot be written as an exterior tensor derivative due to the current terms. In those cases, a formulation similar to that in the next example has to be used. ∎

∎ **Example 2.12** As an example involving the Hodge dual (see computer algebra code [52]), we derive the homogeneous Maxwell equations from a tensor notation containing the Hodge dual of

the electromagnetic field tensor introduced in the preceding example, 2.11. In tensor notation, the equation is:

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{2.185}$$

and involves the Hodge dual of the 4 x 4 field tensor, defined as follows:

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & -E^3 & E^2 \\ cB^2 & E^3 & 0 & -E^1 \\ cB^3 & -E^2 & E^1 & 0 \end{bmatrix}. \tag{2.186}$$

Indices are raised using the Minkowski metric (2.176):

$$\widetilde{F}^{\mu\nu} = \eta^{\mu\kappa}\eta^{\nu\rho}\widetilde{F}_{\kappa\rho}. \tag{2.187}$$

Therefore, the covariant Hodge dual is:

$$\widetilde{F}^{\mu\nu} = \begin{bmatrix} 0 & cB^1 & cB^2 & cB^3 \\ -cB^1 & 0 & -E^3 & E^2 \\ -cB^2 & E^3 & 0 & -E^1 \\ -cB^3 & -E^2 & E^1 & 0 \end{bmatrix}, \tag{2.188}$$

for example:

$$\widetilde{F}_{01} = \frac{1}{2}(\varepsilon_{0123}F^{23} + \varepsilon_{0132}F^{32}) = F^{23} \tag{2.189}$$

and

$$\widetilde{F}^{01} = \eta^{00}\eta^{11}\widetilde{F}_{01} = -\widetilde{F}_{01}. \tag{2.190}$$

The homogeneous laws of classical electrodynamics are obtained as follows, by choice of indices. The Gauss law is obtained by choosing:

$$\nu = 0 \tag{2.191}$$

and so

$$\partial_1 \widetilde{F}^{10} + \partial_2 \widetilde{F}^{20} + \partial_3 \widetilde{F}^{30} = 0. \tag{2.192}$$

In vector notation this is

$$\nabla \cdot \mathbf{B} = 0. \tag{2.193}$$

The Faraday law of induction is obtained by choosing:

$$\nu = 1,2,3 \tag{2.194}$$

and consists of three component equations:

$$\begin{aligned} &\partial_0 \widetilde{F}^{01} + \partial_2 \widetilde{F}^{21} + \partial_3 \widetilde{F}^{31} = 0 \\ &\partial_0 \widetilde{F}^{02} + \partial_1 \widetilde{F}^{12} + \partial_3 \widetilde{F}^{32} = 0 \\ &\partial_0 \widetilde{F}^{03} + \partial_1 \widetilde{F}^{13} + \partial_2 \widetilde{F}^{23} = 0. \end{aligned} \tag{2.195}$$

These can be condensed into one vector equation, which is

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0. \tag{2.196}$$

The differential form, tensor and vector notations are summarized as follows:

$$d \wedge F = 0 \rightarrow \partial_\mu \widetilde{F}^{\mu\nu} = 0 \rightarrow \nabla \cdot \mathbf{B} = 0 \tag{2.197}$$
$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0.$$

The homogeneous laws of classical electrodynamics are most elegantly represented by the differential form notation, but most usefully represented by the vector notation. ∎

Exterior covariant derivative

So far, we have seen that exterior derivatives are antisymmetric sums of partial derivatives applied to n-forms. The question now is what happens if we want to combine the concept of the exterior derivative with a covariant derivative. This is a generalization of the concept, which should be more appropriate to curved manifolds where covariant derivatives play an important role for their description, for example, to define commutators as in Section 2.4.2. We can define an exterior covariant derivative by creating an (n+1)-form from an n-form A:

$$D \wedge A := (D \wedge A)_{\mu_1 \ldots \mu_{n+1}} = D_{[\mu} \wedge A_{\nu_1 \cdots \nu_n]}. \tag{2.198}$$

For a 1-form A_ν this then is

$$\begin{aligned} D \wedge A = (D \wedge A)_{\mu\nu} &= D_{[\mu} \wedge A_{\nu]} = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda \\ &= \partial_{[\mu} A_{\nu]} - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) A_\lambda \end{aligned} \tag{2.199}$$

and with the definition (2.132) of the torsion tensor this can be written:

$$D \wedge A = \partial_{[\mu} A_{\nu]} - T^\lambda{}_{\mu\nu} A_\lambda. \tag{2.200}$$

Since the right-hand side is a tensor, $D \wedge A$ is also a tensor. The equation can be written in form notation:

$$D \wedge A = d \wedge A - T\, A. \tag{2.201}$$

We will extend this concept further in the next chapter.

2.5 Cartan geometry

Having developed the basics of Riemannian geometry, including torsion, we now approach the central point of this book: Cartan geometry. This will be the mathematical foundation of all fields of physics, as we will see.

2.5.1 Tangent space, tetrads and metric

By using Riemannian geometry as a basis, we have available nearly all tools that we need to develop the geometry that is called Cartan geometry and is the basis of ECE theory. We now need to set our focus on tangent spaces. In Section 2.1 we dealt with coordinate transformations in the base manifold. The tangent space at a point x in the base manifold was introduced as a Minkowski space of the same dimension for the local neighborhood of x. A vector V^μ defined in the base manifold can be transformed to a vector in tangent space denoted by V^a. We introduce Latin indices to

denote vectors and tensors in tangent space. A vector in the base manifold can be transformed to the corresponding one in the tangent space by a transformation matrix q. This is similar to introduction of the transformation matrix α in Eqs. (2.46 ff.), but with the difference that the transformation takes place between two different spaces. The basic transformation is

$$V^a = q^a{}_\mu V^\mu \tag{2.202}$$

with transformation matrix elements $q^a{}_\mu$. This is the basis of Cartan geometry, and q is called the *tetrad*. q transforms between the base manifold and tangent space. The inverse transformation is $q^{-1} = (q^\mu{}_a)$, producing a vector in the base manifold:

$$V^\mu = q^\mu{}_a V^a. \tag{2.203}$$

If the metric of the tangent space η_{ab} is transformed to the base manifold (this is a (0,2) tensor), the result must be the metric of the base manifold $g_{\mu\nu}$ by definition:

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu\, \eta_{ab}, \tag{2.204}$$

and inversely:

$$\eta_{ab} = \frac{1}{n}\, q^\mu{}_a q^\nu{}_b\, g_{\mu\nu}, \tag{2.205}$$

where n is the dimension of the base manifold. Since q is a coordinate transformation, the product of q and its inverse has to be the unit matrix:

$$\mathbf{q}\mathbf{q}^{-1} = \mathbf{1} \tag{2.206}$$

which, written in component form, is

$$q^a{}_\mu q^\nu{}_a = \delta^\nu_\mu, \tag{2.207}$$
$$q^a{}_\mu q^\mu{}_b = \delta^a_b. \tag{2.208}$$

The sum of the diagonal elements of (2.206), called the *trace*, is the dimension of the spaces between which the transformation takes place:

$$q^a{}_\mu q^\mu{}_a = n. \tag{2.209}$$

However, this kind of summed product will often occur in our calculations and it is beneficial to let the result be unity:

$$q^a{}_\mu q^\mu{}_a := 1. \tag{2.210}$$

Therefore, we introduce a scaling factor of $1/\sqrt{n}$ to the tetrad elements and $\sqrt{n}$ to the inverse tetrad elements:

$$q^a{}_\mu \to \frac{1}{\sqrt{n}}\, q^a{}_\mu, \tag{2.211}$$
$$q^\mu{}_a \to \sqrt{n}\, q^\mu{}_a. \tag{2.212}$$

Thus, the conditions (2.204) and (2.205) remain satisfied.

■ **Example 2.13** We consider the transformation to spherical polar coordinates, Eq. (2.58) from Example (2.4):

$$\alpha = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.213}$$

The inverse transformation is

$$\alpha^{-1} = \begin{bmatrix} \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta) \\ \frac{\cos(\phi)\cos(\theta)}{r} & \frac{\sin(\phi)\cos(\theta)}{r} & -\frac{\sin(\theta)}{r} \\ -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{bmatrix} \tag{2.214}$$

as can be seen from computer algebra code [53]. To make this transformation a tetrad from a cartesian base manifold to a Euclidian tangent space with spherical polar coordinates, we have to set

$$\mathbf{q} = \frac{1}{\sqrt{3}}\,\alpha, \tag{2.215}$$

$$\mathbf{q}^{-1} = \sqrt{3}\,\alpha^{-1}. \tag{2.216}$$

Then we have

$$\mathbf{q}\mathbf{q}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.217}$$

which is the unit matrix as required. ■

2.5.2 Derivatives in tangent space

We will now investigate the differential calculus in tangent space and how it is connected to that of the base manifold. The tangent space at a point x is a Euclidian space, and we could argue that this allows us to use ordinary differentiation. To define a derivative, we have to construct infinitesimal transitions from the neighborhood of x. For a point $y \neq x$, however, another tangent space is defined because of the definition of tangent spaces. Therefore, the curved structure of the base manifold has to be respected in the definition of the derivatives in tangent space. In the base manifold, we defined the covariant derivative for this purpose, see Eq. (2.120):

$$D_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{2.218}$$

where the partial derivatives ∂_μ and the connection coefficients $\Gamma^\nu_{\mu\lambda}$ are operating on a vector V^λ in the base manifold. We can do the same definition for a vector V^a in tangent space, but the connection coefficients are different here:

$$D_\mu V^a := \partial_\mu V^a + \omega^a{}_{\mu b} V^b. \tag{2.219}$$

The role of the connection coefficients is taken over by other coefficients called *spin connections* $\omega^a{}_{\mu b}$. These have the same number of indices as the Γ's but transform in the tangent space. Therefore they have two Latin indices. The name "spin connection" comes from the fact that this can be used to define covariant derivatives of spinors, which is actually impossible using the Γ connection coefficients. The derivative D_μ itself is defined with respect to the base manifold and therefore has a Greek index. This also has to be present in the spin connection to maintain the indices as required for a tensor expression.

Covariant derivatives of a mixed index tensor are defined in a way so that the indices of tangent space are accompanied by a spin connection and the indices of the base manifold by a Christoffel connection, for example:

$$D_\mu V^a{}_\nu = \partial_\mu V^a{}_\nu + \omega^a{}_{\mu b} V^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} V^a{}_\lambda \,. \tag{2.220}$$

or

$$D_\mu X^{ab}{}_{c\nu} = \partial_\mu X^{ab}{}_{c\nu} + \omega^a{}_{\mu d} X^{db}{}_{c\nu} + \omega^b{}_{\mu d} X^{ad}{}_{c\nu} - \omega^d{}_{\mu c} X^{ab}{}_{d\nu} - \Gamma^\lambda{}_{\mu\nu} X^{ab}{}_{c\lambda} \,. \tag{2.221}$$

In the second example d and λ are dummy indices. The summations over lower (contravariant) indices have a minus sign for both the spin connection and Christoffel connection terms. The spin connections are not tensors, as this holds for the Γ connections. However, the expressions with covariant derivatives are tensors.

2.5.3 Exterior derivatives in tangent space

In section 2.4.3 we introduced exterior derivatives. These are n-forms based on covariant derivatives. Considering a mixed-index tensor V^a_μ, we can interpret this as a vector-valued 1-form where a is the index of the vector component. So V^a would be a short notation of this 1-form. The concept of antisymmetric n-forms has been introduced in section 2.3. An exterior derivative of n-forms has been introduced in section 2.4.3, where a p-form is extended to a (p+1)-form by introducing the antisymmetric derivative operator $d\wedge$, see Eq. (2.167):

$$(d\wedge A)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}. \tag{2.222}$$

We can extend this concept to the tangent space. First, the definition of the covariant derivative can be extended to mixed-index tensors by giving A one or more indices of tangent space:

$$(d\wedge A^b)_{\mu_1\ldots\mu_{p+1}} := (p+1)\partial_{[\mu_1} A^b{}_{\mu_2\ldots\mu_{p+1}]}. \tag{2.223}$$

This definition stands on its own, but in curved manifolds it becomes important to define a covariant exterior derivative of p-forms by basing this definition on the covariant derivative operator D_μ. In form notation, this kind of covariant derivative is written as

$$(D\wedge A^b)_{\mu_1\ldots\mu_{p+1}} := (p+1) D_{[\mu_1} A^b{}_{\mu_2\ldots\mu_{p+1}]}. \tag{2.224}$$

where the D's at the right-hand side are the "usual" covariant derivatives of coordinate index μ_1, etc., as defined in (2.220) for example. A may be a tensor of an arbitrary number of Greek and Latin indices, as before. The lower Greek indices define the p-form. In short indexless notation we can also write:

$$D\wedge A := (p+1) D_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}. \tag{2.225}$$

We will come back to this short-hand notation later. For example, Eq. (2.220) with exterior covariant derivative and coordinate indices $\mu \in \{0,1,2\}$ reads:

$$\begin{aligned} D\wedge V^a &= (D\wedge V^a)_{\mu\nu} \\ &= 2\,(D_0 V^a_1 + D_1 V^a_2 + D_2 V^a_0 - D_1 V^a_0 - D_2 V^a_1 - D_0 V^a_2\,) \\ &= 2\,(D_0(V^a_1 - V^a_2\,) + D_1(V^a_2 - V^a_0\,) + D_2(V^a_0 - V^a_1\,)) \end{aligned} \tag{2.226}$$

where the "normal" covariant derivatives are defined as before, for example:

$$D_0 V^a_1 = \partial_0 V^a_1 + \omega^a{}_{0b} V^b_1 - \Gamma^\lambda{}_{01} V^a_\lambda \,. \tag{2.227}$$

The antisymmetry of the 2-form (2.226) requires

$$(D\wedge V^a)_{\mu\nu} = -(D\wedge V^a)_{\nu\mu} \tag{2.228}$$

from which it follows that interchanging the indices μ and ν gives the negative result of (2.226). That this is the case can be seen directly from the second line of the equation.

2.5.4 Tetrad postulate

Since the tangent space is uniquely related to the base manifold via the tetrad matrix $q^a{}_\mu$, the Γ-connections of the base manifold and spin connections of the tangent space are related to each other. To see how this is the case, we use the so-called *metric compatibility*, the statement that a vector must be the same when described in different coordinate systems. This is necessary for physical uniqueness, otherwise we would be dealing with a kind of mathematics that is not related to physical objects and processes. We introduced this concept in Section 2.4.2 for vectors in the base manifold, and here we extend it to the tangent space in Cartan geometry.

Having this in mind, we can represent a covariant derivative of a tangent vector in two different ways. Denoting the orthonormal unit vectors in the base manifold by $\hat{e}_\nu$ and those of the tangent space by $\hat{e}_a$, we can write

$$DV = D_\mu V^\nu = (\partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda)\hat{e}_\nu \tag{2.229}$$

and

$$DV = D_\mu V^a = (\partial_\mu V^a + \omega^a{}_{\mu b} V^b)\hat{e}_a \tag{2.230}$$

for the same vector DV. In the latter case, one also speaks of a *mixed basis* because the derivative relates to the manifold as before. The latter equation can be transformed into the base manifold coordinates by transforming the coordinates V^a and the unit vectors $\hat{e}_a$ according to the rules introduced in Section 2.5.1 and with renaming of dummy indices:

$$\begin{aligned} D_\mu V^a &= \left(\partial_\mu V^a + \omega^a{}_{\mu b} V^b\right)\hat{e}_a \\ &= \left(\partial_\mu (q^a{}_\nu V^\nu) + \omega^a{}_{\mu b} q^b{}_\lambda V^\lambda\right) q^\sigma{}_a \hat{e}_\sigma \\ &= q^\sigma{}_a \left(q^a{}_\nu \partial_\mu V^\nu + V^\nu \partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\lambda V^\lambda\right)\hat{e}_\sigma \\ &= \left(\partial_\mu V^\nu + q^\nu{}_a V^\lambda \partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^\nu{}_a q^b{}_\lambda V^\lambda\right)\hat{e}_\nu . \end{aligned} \tag{2.231}$$

Comparing with Eq. (2.229) then directly gives

$$\boxed{\Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \partial_\mu q^a{}_\lambda + q^\nu{}_a q^b{}_\lambda \omega^a{}_{\mu b} .} \tag{2.232}$$

Multiplying this equation with $q^\lambda{}_c$ and applying the same rules as above gives

$$q^\lambda{}_c \Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \omega^a{}_{\mu c} + q^\lambda{}_c q^\nu{}_a \partial_\mu q^a{}_\lambda \tag{2.233}$$

and multiplying with $q^b{}_\nu$ gives

$$q^b{}_\nu q^\lambda{}_c \Gamma^\nu{}_{\mu\lambda} = \omega^b{}_{\mu c} + q^\lambda{}_c \partial_\mu q^b{}_\lambda , \tag{2.234}$$

which after renaming of indices is

$$\boxed{\omega^a{}_{\mu b} = q^a{}_\nu q^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - q^\lambda{}_b \partial_\mu q^a{}_\lambda .} \tag{2.235}$$

Thus, we have obtained the relations between both types of connections that we needed. Knowing one of them and the tetrad matrix allows us to compute the other connection.

We can further multiply Eq. (2.232) by $q^c{}_\nu$, obtaining (after applying the rules)

$$q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\lambda + q^b{}_\lambda \omega^a{}_{\mu b} . \tag{2.236}$$

As can be seen by comparison with (2.220), these are exactly the terms of the covariant derivative of the tensor $q^a{}_\nu$ in a mixed basis. It follows that

$$\boxed{D_\mu q^a{}_\nu = 0.} \tag{2.237}$$

This is called the *tetrad postulate*. It states that the covariant derivative of all tetrad elements vanishes.

This is a consequence of metric compatibility, which we postulated at the beginning of this section. As was shown earlier in Eq. (2.139), metric compatibility in the base manifold is defined by an analogue equation for the metric:

$$D_\sigma g_{\mu\nu} = 0. \tag{2.238}$$

If the space is Euclidean, we have

$$D_\sigma \eta_{\mu\nu} = 0 \tag{2.239}$$

for the Minkowski metric (2.176). Since this is also the metric for the tangent space, we can apply the corresponding definition of the covariant derivative:

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega^c{}_{\mu a}\,\eta_{cb} - \omega^c{}_{\mu b}\,\eta_{ac} = 0. \tag{2.240}$$

The Minkowski metric lowers the Latin indices of the spin connections so that we have

$$-\omega_{a\mu b} - \omega_{b\mu a} = 0 \tag{2.241}$$

or

$$\omega_{a\mu b} = -\omega_{b\mu a}\,. \tag{2.242}$$

Metric compatibility provides the property of antisymmetry for the spin connections. Notice that antisymmetry is only defined if the respective indices are all at the lower or upper positon. Despite this antisymmetry, the spin connection is not a tensor, as is also the case for the Γ connection. The symmetry properties of the Γ connection were discussed in Section 2.4.2.

■ **Example 2.14** We compute some spin connection examples from Eq. (2.235). We need a given geometry defined by a tetrad and the Christoffel connection coefficients. We will use example 2.13, where we considered a transformation to spherical polar coordinates. We interpret this in such a way that the polar coordinates of the base manifold are transformed into cartesian coordinates of the tangent space. According to Eqs. (2.213) and (2.215) the tetrad matrix then is

$$\mathbf{q} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.243}$$

The spin connections for spherical polar coordinates have been investigated in three variants in example 2.10:

1. a general connection,
2. a connection antisymmetrized in the non-diagonal lower indices,
3. a symmetric connection (used in Einsteinian relativity).

These functions for the Γ's have to be inserted into Eq. (2.235), together with the tetrad elements of (2.243). Please notice that both the tetrad and inverse tetrad elements occur in (2.235). The $q^a{}_\nu$ are the elements of (2.243) and the $q^\nu{}_a$ are those of the inverted tetrad matrix, essentially Eq. (2.214).

The calculation is lengthy and has been automated through computer algebra code [54]. The results for case 1 (the general connection) are, for example:

$$\begin{aligned}
\omega^{(1)}{}_{1(1)} &= 0, \\
\omega^{(1)}{}_{1(2)} &= -\sin(\theta)\,(A_9\, r\sin(\theta) + A_8\cos(\theta)), \\
\omega^{(1)}{}_{1(3)} &= A_8\sin(\phi)\sin(\theta)^2 - A_9\sin(\phi)\, r\cos(\theta)\sin(\theta) - \frac{A_7\cos(\phi)}{r}.
\end{aligned} \tag{2.244}$$

The A's are constants contained in the Γ's. Obviously they have different physical units, otherwise there would be problems in summation. In order to make it easier to distinguish between Latin and Greek indices, the numbers for Latin indices have been set in parentheses. For case 2 (above), the results are simpler:

$$\begin{aligned}
\omega^{(1)}{}_{1(1)} &= 0, \\
\omega^{(1)}{}_{1(2)} &= \frac{A_{10}\cos(\theta)}{r^2\sin(\theta)}, \\
\omega^{(1)}{}_{1(3)} &= -\frac{A_{10}\sin(\phi)}{r^2},
\end{aligned} \tag{2.245}$$

and in case 3 (symmetric Christoffel connections), all spin connections vanish:

$$\omega^{a}{}_{\mu b} = 0, \tag{2.246}$$

indicating that there is no spin connection for a geometry without torsion. The antisymmetry holds even for the case where a and b are indices at different positions (upper and lower), because the metric in tangent space is the unit matrix. The antisymmetry has been checked using the code, and it is always

$$\omega^{a}{}_{\mu b} = -\omega^{b}{}_{\mu a} \tag{2.247}$$

as required. ∎

2.5.5 Evans lemma

We now come to some more specifically relevant properties of Cartan geometry. The tetrad postulate can be modified to give a differential equation of second order for the tetrad elements. This equation is a wave equation and is fundamental for many fields of physics. The tetrad postulate (2.237) can be augmented by an additional derivative:

$$D^{\mu}(D_{\mu}q^{a}{}_{\nu}) = 0. \tag{2.248}$$

We introduced a covariant derivative with upper index in order to make μ a summation (dummy) index. Because the expression in the parentheses is a scalar function due to the tetrad postulate, we need not bother with how this derivative is defined, it reduces to a partial derivative by definition. So, we can write:

$$\partial^{\mu}(D_{\mu}q^{a}{}_{\nu}) = 0. \tag{2.249}$$

or

$$\partial^{\mu}(\partial_{\mu}q^{a}{}_{\nu} + \omega^{a}{}_{\mu b}q^{b}{}_{\nu} - \Gamma^{\lambda}{}_{\mu\nu}q^{a}{}_{\lambda}) = 0. \tag{2.250}$$

In a manifold with 4-vectors $[ct, X, Y, Z]$, the contravariant form of the partial derivative is defined in the usual way:

$$[\partial_0, \partial_1, \partial_2, \partial_3] = \left[\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], \tag{2.251}$$

while the covariant form of the partial derivative is defined with sign changed for the spatial derivatives:

$$\left[\partial^0, \partial^1, \partial^2, \partial^3\right] = \left[\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial X}, -\frac{\partial}{\partial Y}, -\frac{\partial}{\partial Z}\right]. \tag{2.252}$$

Therefore $\partial^\mu \partial_\mu$ is the d'Alambert operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2}. \tag{2.253}$$

Then from Eq. (2.250) follows

$$\Box\, q^a{}_\nu + G^a{}_\nu = 0 \tag{2.254}$$

a wave equation with the tensor function

$$G^a{}_\nu = \partial^\mu(\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu(\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda). \tag{2.255}$$

This equation can be made an eigenvalue equation by requiring $G^a{}_\nu$ to be split into a tetrad part and a scalar function R:

$$G^a{}_\mu = R\, q^a{}_\nu \tag{2.256}$$

with

$$R = q^\nu{}_a \left(\partial^\mu(\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu(\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda)\right). \tag{2.257}$$

R contains only dummy indices and is a scalar function. Then (2.254) can be written as

$$\boxed{\Box\, q^a{}_\nu + R\, q^a{}_\nu = 0} \tag{2.258}$$

and is called the *Evans lemma*. It is a generally covariant eigenvalue equation. R plays the role of a curvature, as we will see in later chapters. The entire field of generally covariant quantum mechanics is based on this equation. The equation is highly non-linear, because R depends on the eigenfunction $q^a{}_\nu$ and the Christoffel and spin connections. In later chapters, in a first approximation, we will often assume that R is a constant.

2.5.6 Maurer-Cartan structure equations

The torsion and curvature tensors of Riemannian geometry can be transformed to 2-forms of Cartan geometry simply by defining

$$T^a{}_{\mu\nu} := q^a{}_\kappa T^\kappa{}_{\mu\nu}\,, \tag{2.259}$$

$$R^a{}_{b\mu\nu} := q^a{}_\rho q^\sigma{}_b R^\rho{}_{\sigma\mu\nu}. \tag{2.260}$$

Multiplication with tetrad elements replaces some Greek indices with Latin indices of the tangent space, so the torsion and curvature tensors defined in Eqs. (2.132) and (2.134) are made 2-forms of

torsion and curvature. To these forms two foundational relations apply, which will be derived in this section, using the proof described in [11].

We first define forms of the Christoffel and spin connections similarly to (2.259) and (2.260):

$$\Gamma^{a}{}_{\mu\nu} := q^{a}{}_{\lambda}\,\Gamma^{\lambda}{}_{\mu\nu}, \tag{2.261}$$

$$\omega^{a}{}_{\mu\nu} := q^{b}{}_{\nu}\,\omega^{a}{}_{\mu b}. \tag{2.262}$$

These are both 2-forms as well. The tetrad postulate (2.237) can be formulated by inserting these definitions into (2.236):

$$\Gamma^{a}{}_{\mu\nu} = \partial_{\mu} q^{a}{}_{\nu} + \omega^{a}{}_{\mu\nu}. \tag{2.263}$$

Inserting the definition of torsion

$$T^{\kappa}{}_{\mu\nu} := \Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\nu\mu} \tag{2.264}$$

into (2.259) gives

$$T^{a}{}_{\mu\nu} = q^{a}{}_{\kappa}(\Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\nu\mu}) = \Gamma^{a}{}_{\mu\nu} - \Gamma^{a}{}_{\nu\mu}, \tag{2.265}$$

and inserting relation (2.263) gives

$$T^{a}{}_{\mu\nu} = \partial_{\mu} q^{a}{}_{\nu} - \partial_{\nu} q^{a}{}_{\mu} + \omega^{a}{}_{\mu\nu} - \omega^{a}{}_{\nu\mu}. \tag{2.266}$$

This can be written with the $\wedge$ operator for antisymmetric forms, introduced in Example 2.8 and Section 2.4.3 as

$$(T^{a})_{\mu\nu} = (d \wedge q^{a})_{\mu\nu} + (\omega^{a}{}_{b} \wedge q^{b})_{\mu\nu} \tag{2.267}$$

or, in short form notation:

$$\boxed{T^{a} = d \wedge q^{a} + \omega^{a}{}_{b} \wedge q^{b}} \tag{2.268}$$

which is called the *first Maurer-Cartan structure equation*.

The Riemann curvature tensor is defined

$$R^{\lambda}{}_{\rho\mu\nu} := \partial_{\mu}\Gamma^{\lambda}_{\nu\rho} - \partial_{\nu}\Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}. \tag{2.269}$$

We define additional 1-forms of the Christoffel connection:

$$\Gamma^{a}{}_{\mu b} := q^{a}{}_{\lambda} q^{\nu}{}_{b} \Gamma^{\lambda}{}_{\mu\nu} \tag{2.270}$$

and from (2.263) we have

$$\Gamma^{a}{}_{\mu b} = q^{\nu}{}_{b}(\partial_{\mu} q^{a}{}_{\nu} + \omega^{a}{}_{\mu\nu}). \tag{2.271}$$

Then the curvature form (2.260) can be written:

$$R^{a}{}_{b\mu\nu} = \partial_{\mu}\Gamma^{a}{}_{\nu b} - \partial_{\nu}\Gamma^{a}{}_{\mu b} + \Gamma^{a}{}_{\mu c}\Gamma^{c}{}_{\nu b} - \Gamma^{a}{}_{\nu c}\Gamma^{c}{}_{\mu b}. \tag{2.272}$$

This is an antisymmetric 2-form that in form notation reads:

$$R^{a}{}_{b} = d \wedge \Gamma^{a}{}_{b} + \Gamma^{a}{}_{c} \wedge \Gamma^{c}{}_{b}. \tag{2.273}$$

The first term on the right-hand side is

$$d \wedge \Gamma^a{}_b = (d \wedge d \wedge q^a) q_b + d \wedge \omega^a{}_b = d \wedge \omega^a{}_b \tag{2.274}$$

because of the rule $d \wedge d \wedge a = 0$ for any form a. The second term of (2.273) is

$$\Gamma^a{}_c \wedge \Gamma^c{}_b = (q_c\, d \wedge q^a + \omega^a{}_c) \wedge (q_b\, d \wedge q^c + \omega^c{}_b). \tag{2.275}$$

The terms with the exterior derivative can be written with full indices as $q^\nu{}_c \partial_\mu q^a{}_\nu$, for example. The product is summed over by the dummy index ν.

From the Leibniz rule we find:

$$q^\lambda{}_c \partial_\mu q^a{}_\lambda + q^a{}_\lambda \partial_\mu q^\lambda{}_c = \partial_\mu (q^\lambda{}_c q^a{}_\lambda) = \partial_\mu \delta^a_c = 0, \tag{2.276}$$

therefore:

$$q^\lambda{}_c \partial_\mu q^a{}_\lambda = -q^a{}_\lambda \partial_\mu q^\lambda{}_c. \tag{2.277}$$

The summation on the left-hand and right-hand side can be contracted to functions

$$q^a{}_c = -q^a{}_c. \tag{2.278}$$

It follows

$$q^a{}_c = 0, \tag{2.279}$$

$$q^\nu{}_c \partial_\mu q^a{}_\nu = 0. \tag{2.280}$$

Therefore from (2.275):

$$\Gamma^a{}_c \wedge \Gamma^c{}_b = \omega^a{}_c \wedge \omega^c{}_b, \tag{2.281}$$

and with (2.274), we obtain from (2.273):

$$\boxed{R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b} \tag{2.282}$$

which is called the *second Maurer-Cartan structure equation*. Using the definition of the exterior covariant derivative (2.198), the Maurer-Cartan structure equations can be written in the form

$$\boxed{\begin{aligned} T^a &= D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b, \qquad (2.283)\\ R^a{}_b &= D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \qquad (2.284) \end{aligned}}$$

■ **Example 2.15** The validity of structure equations is demonstrated by an example of the transformation to spherical polar coordinates again. The tetrad was defined in Example 2.13, and the spin connections in Example 2.14. For the Gamma connections two versions were used: a general, asymmetric connection, and an antisymmetrized connection, as described in Example 2.14. If we know the Gamma connection, we can compute the torsion form:

$$T^a{}_{\nu\mu} = q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \right) \tag{2.285}$$

and the Riemann form:

$$R^a{}_{b\mu\nu} = q^a{}_\sigma q^\mu{}_b \left(\partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} \right). \tag{2.286}$$

This has been done using computer algebra code [55]. The antisymmetry of the form elements in the two last indexes is checked:

$$T^{a}{}_{\mu\nu} = -T^{a}{}_{\nu\mu}\,, \tag{2.287}$$

$$R^{a}{}_{b\mu\nu} = -R^{a}{}_{b\nu\mu}. \tag{2.288}$$

For example, we find with the antisymmetrized connection:

$$T^{(2)}{}_{11} = 0 \tag{2.289}$$

$$T^{(2)}{}_{13} = \frac{2\cos(\phi)\sin(\theta)}{\sqrt{3}} + \frac{2A_{10}\sin(\phi)\cos(\theta)}{\sqrt{3}r} \tag{2.290}$$

$$T^{(2)}{}_{31} = -\frac{2\cos(\phi)\sin(\theta)}{\sqrt{3}} - \frac{2A_{10}\sin(\phi)\cos(\theta)}{\sqrt{3}r} \tag{2.291}$$

$$R^{(1)}{}_{(3)11} = 0 \tag{2.292}$$

$$R^{(1)}{}_{(3)13} = -\frac{{A_{10}}^{2}\sin(\phi)\cos(\theta)}{r^{3}\sin(\theta)} \tag{2.293}$$

$$R^{(1)}{}_{(3)31} = \frac{{A_{10}}^{2}\sin(\phi)\cos(\theta)}{r^{3}\sin(\theta)} \tag{2.294}$$

Now all elements of the torsion and curvature form are computed, and we are ready to evaluate the right-hand sides of the structure equations (2.283) and (2.284), which in indexed form can be written:

$$D_\mu q^{a}{}_{\nu} - D_\nu q^{a}{}_{\mu} = \partial_\mu q^{a}{}_{\nu} - \partial_\nu q^{a}{}_{\mu} + \omega^{a}{}_{\mu b} q^{b}{}_{\nu} - \omega^{a}{}_{\nu b} q^{b}{}_{\mu} \tag{2.295}$$

and

$$D_\mu \omega^{a}{}_{\nu b} - D_\nu \omega^{a}{}_{\mu b} = \partial_\mu \omega^{a}{}_{\nu b} - \partial_\nu \omega^{a}{}_{\mu b} + \omega^{a}{}_{\mu c}\,\omega^{c}{}_{\nu b} - \omega^{a}{}_{\nu c}\,\omega^{c}{}_{\mu b}\,. \tag{2.296}$$

The covariant derivatives have been resolved according to their definitions for each permutation of (μ,ν). When the indices run over *all* values 1,2, this does not matter because the antisymmetry property sets all quantities with equal indices, for example $(\mu,\nu) = (1,1)$, to zero. In computer algebra code [55], it is shown that the right-hand sides of the structure equations are equal to the definitions of the torsion and curvature form defined by (2.285) and (2.286). In addition, it is shown that re-computing the torsion and curvature tensors from their 2-forms gives the original tensors (2.264) and (2.269):

$$T^{\rho}{}_{\mu\nu} = q^{\rho}{}_{a} T^{a}{}_{\mu\nu}\,, \tag{2.297}$$

$$R^{\sigma}{}_{\rho\mu\nu} = q^{\sigma}{}_{a}\, q^{b}{}_{\rho}\, R^{a}{}_{b\mu\nu}. \tag{2.298}$$

■

3. The fundamental theorems of Cartan geometry

We have now arrived at a knowledge level in Cartan geometry that allows us to formulate the fundamental theorems of this geometry. Some of them are known for a longer time and have been mentioned in textbooks [12], but others have been found during the development of ECE theory. The theorems can easily be formulated in form notation but for the proofs we have to descend to the tensor notation and then climb to the form notation again.

3.1 Cartan-Bianchi identity

The first theorem is called *Cartan-Bianchi identity* [12] and is known as the *first Bianchi identity* or simply the *Bianchi identity* in Riemannian geometry without torsion. We have added the name of Cartan to stress that this theorem connects torsion and curvature in Cartan geometry. In form notation, it reads:

$$D \wedge T^a = R^a{}_b \wedge q^b. \tag{3.1}$$

This is an equation of 3-forms. To prove this equation, we recast the left-hand side into the right-hand side. Inserting the definition of the exterior covariant derivative gives, for the left-hand side:

$$(D \wedge T^a)_{\mu\nu\rho} = (d \wedge T^a)_{\mu\nu\rho} + (\omega^a{}_b \wedge T^b)_{\mu\nu\rho}. \tag{3.2}$$

Since this is an antisymmetric 3-form, we can write in commutator notation (see Section 2.3):

$$D_{[\mu} T^a{}_{\nu\rho]} = \partial_{[\mu} T^a{}_{\nu\rho]} + \omega^a{}_{[\mu b} T^b{}_{\nu\rho]}\,. \tag{3.3}$$

In Example 2.8, we had seen that the six index permutations of a 3-form can be reduced to three cyclic permutations of the indices, by use of antisymmetry properties. Therefore, we obtain

$$\begin{aligned} D_{[\mu} T^a{}_{\nu\rho]} =& \partial_\mu T^a{}_{\nu\rho} + \partial_\nu T^a{}_{\rho\mu} + \partial_\rho T^a{}_{\mu\nu} \\ &+ \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} + \omega^a{}_{\rho b} T^b{}_{\mu\nu}\,. \end{aligned} \tag{3.4}$$

Please notice that the lower b index of the spin connection is not included in the permutations, because it is a Latin index of tangent space.

Inserting the definition of torsion

$$T^a{}_{\nu\mu} = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu} = q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) \tag{3.5}$$

then leads to

$$\begin{aligned} D_{[\mu}T^a{}_{\nu\rho]} =& \partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] + \partial_\nu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right)\right] \\ &+ \partial_\rho \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right)\right] \\ &+ \omega^a{}_{\mu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + \omega^a{}_{\nu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right) \\ &+ \omega^a{}_{\rho b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right). \end{aligned} \tag{3.6}$$

The first term in brackets can be written with help of the Leibniz theorem:

$$\partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] = \left(\partial_\mu q^a{}_\lambda\right) \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \tag{3.7}$$

Applying the tetrad postulate (2.236) in the form

$$\partial_\mu q^a{}_\lambda = q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} - q^b{}_\lambda \omega^a{}_{\mu b}. \tag{3.8}$$

then gives

$$\begin{aligned} \partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] =& \left(q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} - q^b{}_\lambda \omega^a{}_{\mu b}\right) \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \\ &+ q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \end{aligned} \tag{3.9}$$

Adding the first and fourth term of (3.6) causes the terms with $\omega^a{}_{\mu b}$ to cancel out:

$$\begin{aligned} &\partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] + \omega^a{}_{\mu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \\ &= q^a{}_\sigma \Gamma^\sigma{}_{\mu\lambda} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \end{aligned} \tag{3.10}$$

Putting all terms of (3.6) together, we obtain

$$\begin{aligned} &D_{[\mu}T^a{}_{\nu\rho]} = \\ &\quad q^a{}_\sigma \Gamma^\sigma{}_{\mu\lambda} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right) \\ &+ q^a{}_\sigma \Gamma^\sigma{}_{\nu\lambda} \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right) + q^a{}_\lambda \left(\partial_\nu \Gamma^\lambda{}_{\rho\mu} - \partial_\nu \Gamma^\lambda{}_{\mu\rho}\right) \\ &+ q^a{}_\sigma \Gamma^\sigma{}_{\rho\lambda} \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) + q^a{}_\lambda \left(\partial_\rho \Gamma^\lambda{}_{\mu\nu} - \partial_\rho \Gamma^\lambda{}_{\nu\mu}\right). \end{aligned} \tag{3.11}$$

Rearranging the sum:

$$\begin{aligned} &D_{[\mu}T^a{}_{\nu\rho]} = \\ &\quad q^a{}_\lambda \left[\left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho}\right) + q^a{}_\lambda \left(\partial_\nu \Gamma^\lambda{}_{\rho\mu} - \partial_\rho \Gamma^\lambda{}_{\nu\mu}\right) + q^a{}_\lambda \left(\partial_\rho \Gamma^\lambda{}_{\mu\nu} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right)\right] \\ &+ q^a{}_\sigma \left[\Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\rho\nu}\right]. \end{aligned} \tag{3.12}$$

Now, in the first line the dummy index λ is replaced by σ:

$$
\begin{aligned}
&D_{[\mu}T^{a}{}_{\nu\rho]} = \\
&\quad q^{a}{}_{\sigma}\Big[\Big(\partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}{}_{\mu\rho}\Big) + q^{a}{}_{\sigma}\Big(\partial_{\nu}\Gamma^{\sigma}{}_{\rho\mu} - \partial_{\rho}\Gamma^{\sigma}{}_{\nu\mu}\Big) + q^{a}{}_{\sigma}\Big(\partial_{\rho}\Gamma^{\sigma}{}_{\mu\nu} - \partial_{\mu}\Gamma^{\sigma}{}_{\rho\nu}\Big) \\
&+ \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\rho\mu} - \Gamma^{\sigma}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\nu\mu} + \Gamma^{\sigma}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\rho\nu}\Big].
\end{aligned}
\tag{3.13}
$$

This expression can be compared to the definition of the Riemann tensor (2.269) with some renumbering:

$$
R^{\sigma}{}_{\rho\mu\nu} := \partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\rho}. \tag{3.14}
$$

Obviously (3.13) is the cyclic sum of the Riemann tensor:

$$
D_{[\mu}T^{a}{}_{\nu\rho]} = q^{a}{}_{\sigma}\, R^{\sigma}{}_{[\rho\mu\nu]} = q^{a}{}_{\sigma}\, R^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.15}
$$

According to the procedure in Eqs. (2.270-2.273), the Riemann tensor can be written as a 2-Form:

$$
R^{a}{}_{b\nu\rho} = q^{a}{}_{\sigma} q^{\mu}{}_{b}\, R^{\sigma}{}_{\mu\nu\rho}. \tag{3.16}
$$

To bring the right-hand side of (3.15) into this form, we extend the Riemann tensor by a unity term according to rule (2.207):

$$
q^{\tau}{}_{b} q^{b}{}_{\mu} = \delta^{\tau}_{\mu} \tag{3.17}
$$

and re-associate the products:

$$
q^{a}{}_{\sigma}\, R^{\sigma}{}_{\mu\nu\rho} = R^{a}{}_{\mu\nu\rho} = R^{a}{}_{\tau\nu\rho}(q^{\tau}{}_{b}\, q^{b}{}_{\mu})\delta^{\tau}_{\mu} = (R^{a}{}_{\tau\nu\rho} q^{\tau}{}_{b})\, q^{b}{}_{\tau}\delta^{\tau}_{\mu} = R^{a}{}_{b\nu\rho}\, q^{b}{}_{\mu}. \tag{3.18}
$$

Re-introducing the cyclic sum we have

$$
D_{[\mu}T^{a}{}_{\nu\rho]} = q^{b}{}_{[\mu}\, R^{a}{}_{b\nu\rho]} = R^{a}{}_{b[\mu\nu}\, q^{b}{}_{\rho]}. \tag{3.19}
$$

which in form notation gives the Cartan-Bianchi identity:

$$
\boxed{D \wedge T^{a} = R^{a}{}_{b} \wedge q^{b}.} \tag{3.20}
$$

■ **Example 3.1** We check the Cartan-Bianchi identity by computing all required elements according to Example 2.15 (the transformation from cartesian to spherical polar coordinates). The Cartan-Bianchi identity (3.20) can be written in indexed form according to (3.19):

$$
D_{\mu}T^{a}{}_{\nu\rho} + D_{\nu}T^{a}{}_{\rho\mu} + D_{\rho}T^{a}{}_{\mu\nu} = R^{a}{}_{b\mu\nu}\, q^{b}{}_{\rho} + R^{a}{}_{b\nu\rho}\, q^{b}{}_{\mu} + R^{a}{}_{b\rho\mu}\, q^{b}{}_{\nu}. \tag{3.21}
$$

Resolving the covariant derivatives according to (3.4) this finally gives:

$$
\begin{aligned}
&\partial_{\mu}T^{a}{}_{\nu\rho} + \partial_{\nu}T^{a}{}_{\rho\mu} + \partial_{\rho}T^{a}{}_{\mu\nu} + \omega^{a}{}_{\mu b}T^{b}{}_{\nu\rho} + \omega^{a}{}_{\nu b}T^{b}{}_{\rho\mu} + \omega^{a}{}_{\rho b}T^{b}{}_{\mu\nu} \\
&= R^{a}{}_{b\mu\nu}\, q^{b}{}_{\rho} + R^{a}{}_{b\nu\rho}\, q^{b}{}_{\mu} + R^{a}{}_{b\rho\mu}\, q^{b}{}_{\nu}
\end{aligned}
\tag{3.22}
$$

for each index triple (μ,ν,ρ). The left-hand and right-hand sides of this equation are computed using computer algebra code [56], and comparison shows that both sides are equal. We note that this result is obtained for both forms of Gamma connections (unconstrained and symmetrized). The torsion tensor is the same for both forms, but the Gamma and spin connections are different. The Cartan-Bianchi identity holds, irrespective of this difference. ■

3.2 Cartan-Evans identity

In the preceding section, it has been shown that the Cartan-Bianchi identity is a rigorous identity of the Riemannian manifold in which ECE theory is defined. The *Cartan-Evans identity* [13, 14, 15] is a new identity of differential geometry, and is the counterpart of the Cartan-Bianchi identity in dual-tensor representation. Both identities will be identical with the ECE field equations as will be worked out in later chapters. The Cartan-Bianchi identity is valid in the Riemannian manifold, and Cartan geometry in the Riemannian manifold is well known to be equivalent to Riemann geometry, thought to be the geometry of natural philosophy (physics). The same holds for the Cartan-Evans identity, which reads

$$D \wedge \widetilde{T}^a = \widetilde{R}^a{}_b \wedge q^b. \tag{3.23}$$

The concept of the Hodge dual was introduced at the end of Section 2.3, and use of the Hodge dual for Maxwell's equations was already discussed in Example 2.12. In this section, we introduce a Hodge dual connection for use in the covariant Hodge dual derivative. Thereafter, the proof of the Cartan-Evans identity is worked out in full analogy to the proof of the Cartan-Bianchi identity.

As has been seen in previous sections, only the antisymmetric part of the Christoffel connection is essential for Cartan geometry. Restricting the connection to the antisymmetric part, we can define the Hodge dual of the Christoffel connection according to Eq. (2.114) by

$$\Lambda^\lambda{}_{\mu\nu} := \widetilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\beta}, \tag{3.24}$$

where $|g|^{-1/2}$ is the inverse square root of the modulus of the determinant of the metric, a weighting factor, by which the Levi-Civita symbol $\varepsilon_{\alpha\beta\mu\nu}$ is made the totally antisymmetric unit tensor, see section 2.3. In (3.24) the Levi-Civita symbol appears with mixed upper and lower indices. Therefore, we have to raise the first two indices in accordance with (2.115):

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}g^{\rho\alpha}g^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}. \tag{3.25}$$

Since the totally antisymmetric tensor (based on the Levi-Civity symbol) does not change its form for any coordinate transformation, we can use the metric of Minkowski space $\eta_{\mu\nu}$ with $|g| = 1$:

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}. \tag{3.26}$$

In this way, a new connection $\Lambda^\lambda{}_{\mu\nu}$ is defined. It is well known that the connection does not transform as a tensor under the general coordinate transformation, but the antisymmetry in its lower two indices means that its Hodge dual may be defined for each upper index of the connection as in the equation above. The antisymmetry of the connection is the basis for the Cartan-Evans identity, a new and fundamental identity of differential geometry. In ECE theory, it will become the inhomogeneous field equation as was already indicated for the homogeneous Maxwell equation in Example 2.12. Note carefully that the torsion is a tensor, but the connection is not a tensor. The same is true of the Hodge duals of the torsion and connection.

In Eq. (2.147) the fundamental commutator equation of Riemannian geometry was derived:

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\,D_\lambda V^\rho, \tag{3.27}$$

which holds for any vector V^ρ of the base manifold. Now take the Hodge duals of either side of Eq.

(3.27) using:

$$[D_\mu, D_\nu]_{\mathrm{HD}} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}[D_\alpha, D_\beta], \tag{3.28}$$

$$\widetilde{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}R^{\rho}{}_{\sigma\alpha\beta}, \tag{3.29}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}T^{\lambda}{}_{\alpha\beta}. \tag{3.30}$$

Thus:

$$[D_\alpha, D_\beta]_{\mathrm{HD}}V^\rho = \widetilde{R}^{\rho}{}_{\sigma\alpha\beta}V^\sigma - \widetilde{T}^{\lambda}{}_{\alpha\beta}\,D_\lambda V^\rho. \tag{3.31}$$

Re-label indices in Eq. (3.31) to give:

$$[D_\mu, D_\nu]_{\mathrm{HD}}V^\rho = \widetilde{R}^{\rho}{}_{\sigma\mu\nu}V^\sigma - \widetilde{T}^{\lambda}{}_{\mu\nu}\,D_\lambda V^\rho. \tag{3.32}$$

The left-hand side of this equation is defined by:

$$[D_\mu, D_\nu]_{\mathrm{HD}}V^\rho := D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho) \tag{3.33}$$

where the covariant derivatives must be defined by the Hodge dual connection (which was defined in Eq. (3.24)):

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^{\rho}{}_{\mu\lambda}V^\lambda, \tag{3.34}$$

$$D_\nu V^\rho = \partial_\nu V^\rho + \Lambda^{\rho}{}_{\nu\lambda}V^\lambda. \tag{3.35}$$

Working out the algebra of torsion and curvature according to Eqs. (2.132, 2.134):

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \Lambda^{\lambda}{}_{\mu\nu} - \Lambda^{\lambda}{}_{\nu\mu}, \tag{3.36}$$

$$\widetilde{R}^{\lambda}{}_{\mu\nu\rho} = \partial_\mu\Lambda^{\lambda}{}_{\nu\rho} - \partial_\nu\Lambda^{\lambda}{}_{\mu\rho} + \Lambda^{\lambda}{}_{\mu\sigma}\Lambda^{\sigma}{}_{\nu\rho} - \Lambda^{\lambda}{}_{\nu\sigma}\Lambda^{\sigma}{}_{\mu\rho}. \tag{3.37}$$

These are the Hodge dual torsion and curvature tensors of the Riemannian manifold.

Now we prove the Cartan Evans identity as follows. The identity is:

$$D\wedge\widetilde{T}^a = \widetilde{R}^a{}_b\wedge q^b \tag{3.38}$$

or

$$d\wedge\widetilde{T}^a + \omega^a{}_b\wedge\widetilde{T}^b = \widetilde{R}^a{}_b\wedge q^b. \tag{3.39}$$

In tensorial notation, in the Riemannian manifold Eqs. (3.38, 3.39) become:

$$D_\mu\widetilde{T}^a{}_{\nu\rho} + D_\rho\widetilde{T}^a{}_{\mu\nu} + D_\nu\widetilde{T}^a{}_{\rho\mu} = \widetilde{R}^a{}_{\mu\nu\rho} + \widetilde{R}^a{}_{\rho\mu\nu} + \widetilde{R}^a{}_{\nu\rho\mu}, \tag{3.40}$$

which can be written with permutation brackets as

$$D_{[\mu}\widetilde{T}^a{}_{\nu\rho]} = \widetilde{R}^a{}_{[\mu\nu\rho]} = q^a{}_\sigma\,\widetilde{R}^\sigma{}_{[\mu\nu\rho]}. \tag{3.41}$$

This equation is formally identical to (3.15) with the following correspondences:

$$\begin{aligned} &T\to\widetilde{T}, \\ &R\to\widetilde{R}, \\ &\Gamma\to\Lambda. \end{aligned} \tag{3.42}$$

Therefore, the proof of the Cartan-Evans identity can proceed in full analogy to that of the Cartan-Bianchi identity in the previous section. Starting with the equivalent of the left-hand side of Eq. (3.15),

$$D_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} = \partial_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} + \omega^{a}{}_{[\mu b}\widetilde{T}^{b}{}_{\nu\rho]}\,, \tag{3.43}$$

it follows that this expression is equal to its right-hand side equivalent of (3.15):

$$q^{a}{}_{\sigma}\,\widetilde{R}^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.44}$$

It follows the validity of Eqs. (3.40 / 3.41), which are the counterpart of (3.19):

$$D_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} = q^{a}{}_{\sigma}\widetilde{R}^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.45}$$

In form notation, this is the Cartan-Evans identity:

$$\boxed{D\wedge\widetilde{T}^{a} = \widetilde{R}^{a}{}_{b}\wedge q^{b}.} \tag{3.46}$$

In the proof of the Cartan-Bianchi identity, the tetrad postulate (2.237) was used. For the Cartan-Evans identity, this has to be used in the form with the Λ connection:

$$\partial_{\mu}q^{a}{}_{\lambda} + q^{b}{}_{\lambda}\,\omega^{a}{}_{\mu b} - q^{a}{}_{\nu}\Lambda^{\nu}{}_{\mu\lambda} = 0. \tag{3.47}$$

Obviously, here the spin connection ω depends on the Λ connection, not the Γ connection. (It would have been best to use a different symbol for ω, but we stay with ω for convenience.)

In summary, all geometric elements for the Cartan-Evans identity are obtained from the following equation set:

$$\Lambda^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^{\lambda}{}_{\alpha\beta}, \tag{3.48}$$

$$\omega^{a}{}_{\mu b} = q^{a}{}_{\nu}q^{\lambda}{}_{b}\Lambda^{\nu}{}_{\mu\lambda} - q^{\lambda}{}_{b}\partial_{\mu}q^{a}{}_{\lambda}, \tag{3.49}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \Lambda^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\nu\mu}, \tag{3.50}$$

$$\widetilde{R}^{\lambda}{}_{\mu\nu\rho} = \partial_{\mu}\Lambda^{\lambda}_{\nu\rho} - \partial_{\nu}\Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\mu\rho}. \tag{3.51}$$

Alternatively, the Hodge duals of curvature and torsion can be computed from the original quantities (based on the Γ connection):

$$\widetilde{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}R^{\rho}{}_{\sigma\alpha\beta}, \tag{3.52}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}T^{\lambda}{}_{\alpha\beta}. \tag{3.53}$$

The 2-forms of $\widetilde{T}^{a}$ and $\widetilde{R}^{a}{}_{b}$ are obtainable in the usual way by multiplying with tetrad elements:

$$\widetilde{R}^{a}{}_{b\mu\nu} = q^{a}{}_{\rho}q^{\sigma}{}_{b}\widetilde{R}^{\rho}{}_{\sigma\mu\nu}, \tag{3.54}$$

$$\widetilde{T}^{a}{}_{\mu\nu} = q^{a}{}_{\lambda}\widetilde{T}^{\lambda}{}_{\mu\nu}\,. \tag{3.55}$$

One of the novel inferences of the Cartan-Evans identity is that there is a Hodge dual connection in the Riemannian manifold in four dimensions. This is a basic discovery, and may be developed in pure mathematics using any type of manifold. However, that development is not of interest to physics by Ockham's Razor, and the need to test a theory against experimental data.

■ **Example 3.2** In analogy to Example 3.1, we check the Cartan-Evans identity by computing all required elements according to Example 2.15 (the transformation from cartesian to spherical polar coordinates). The Cartan-Evans identity (3.46) can be written in indexed form according to (3.45):

$$D_\mu \widetilde{T}^a{}_{\nu\rho} + D_\nu \widetilde{T}^a{}_{\rho\mu} + D_\rho \widetilde{T}^a{}_{\mu\nu} = \widetilde{R}^a{}_{b\mu\nu}\, q^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}\, q^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}\, q^b{}_\nu. \tag{3.56}$$

Resolving the covariant derivatives according to (3.4) finally gives:

$$\begin{aligned} &\partial_\mu \widetilde{T}^a{}_{\nu\rho} + \partial_\nu \widetilde{T}^a{}_{\rho\mu} + \partial_\rho \widetilde{T}^a{}_{\mu\nu} + \omega^a{}_{\mu b} \widetilde{T}^b{}_{\nu\rho} + \omega^a{}_{\nu b} \widetilde{T}^b{}_{\rho\mu} + \omega^a{}_{\rho b} \widetilde{T}^b{}_{\mu\nu} \\ &= \widetilde{R}^a{}_{b\mu\nu}\, q^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}\, q^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}\, q^b{}_\nu \end{aligned} \tag{3.57}$$

for each index triple (μ, ν, ρ). The left-hand and right-hand sides of this equation are computed using computer algebra code [57, 58]. There is however a difference. While the Cartan-Bianchi identity holds for any dimension n of Riemannian space, introducing the Hodge dual for the Cartan-Evans identity constrains the dimension of the dual 2-forms to $n-2$. So, to obtain comparable equations for both identities, we have to use $n = 4$ in the example, leading to 2-forms of the Hodge duals, as well. We have to extend the transformation matrix α (Eq. (2.213)) by the 0-component (time coordinate), resulting in

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{3.58}$$

The time coordinate remains unaltered by the transformation. The n-dimensional metric tensor $\mathbf{g}$ can be computed from the tetrad by (2.204):

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu \eta_{ab}. \tag{3.59}$$

We obtain the metric tensor:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \tag{3.60}$$

which has the modulus of the determinant

$$|g| = r^4 \sin^2\theta. \tag{3.61}$$

The Levi-Civita symbol $\varepsilon_{\alpha\beta\mu\nu}$ in four dimensions can be computed by the formula

$$\varepsilon_{a_0,a_1,a_2,a_3} = \mathrm{sig}(a_3-a_0)\ \mathrm{sig}(a_3-a_1)\ \mathrm{sig}(a_3-a_2)\ \mathrm{sig}(a_2-a_0)\ \mathrm{sig}(a_2-a_1)\ \mathrm{sig}(a_1-a_0). \tag{3.62}$$

Now we have all of the elements that we need to evaluate Eqs. (3.48-3.51). With these, both sides of Eq. (3.56) can be evaluated as was done in Example 3.1. We do this in two examples, using computer algebra. In the first example we repeat the calculations of example 3.1 (Cartan-Bianchi identity) in four dimensions [57]. An interesting result is that the Gamma connection, obtained with additional antisymmetry conditions, has only 4 free parameters. This is similar to Einstein's theory, where the symmetric metric is only determined up to 4 parameters that can be chosen freely and represent "free choice of coordinates". In Cartan geometry, the metric is uniquely defined from the tetrad. The "free choice" appears in the connections. Therefore, this choice is also present

in torsion and curvature, and finally in the fundamental theorems. Some results from computer algebra code [57] are:

$$\begin{aligned}
\Gamma^{0}{}_{12} &= A_4 r^2 \\
\Gamma^{3}{}_{01} &= -\frac{A_2}{r^2 \sin^2\theta} \\
\omega^{(2)}{}_{1(3)} &= \frac{A_3 \cos\phi}{r^2} \\
T^{2}{}_{13} &= \frac{2A_3}{r^2} \\
R^{0}{}_{213} &= \frac{A_1 r \sin\theta - A_2 \cos\theta}{\sin\theta}
\end{aligned} \tag{3.63}$$

As in the preceding example, comparison of both sides of the Cartan-Bianchi identity shows that both sides are equal, in this case for four dimensions.

In the second computer algebra code [58], the Hodge dual connections Λ and ω and the tensors $\widetilde{T}$, $\widetilde{R}$ and their corresponding 2-forms are computed. We obtain, for example, for the Hodge dual connections and tensors:

$$\begin{aligned}
\Lambda^{0}{}_{03} &= \frac{A_4}{\sin^2\theta} \\
\Lambda^{3}{}_{01} &= \frac{\cos\theta}{r^2 \sin^3\theta} \\
\omega^{(2)}{}_{1(3)} &= \frac{\sin\phi\,\left(r^2 \cos\theta \sin^2\theta + A_2 \sin\theta - A_1 r \cos\theta\right)}{r^3 \sin\theta} \\
\widetilde{T}^{2}{}_{13} &= 0 \\
\widetilde{T}^{2}{}_{02} &= -\frac{2A_3}{r^4 \sin^2\theta} \\
\widetilde{R}^{0}{}_{213} &= 0 \\
\widetilde{R}^{0}{}_{202} &= -\frac{2A_2 r^2 \cos\theta \sin\theta - {A_2}^2}{r^4 \sin^4\theta}
\end{aligned} \tag{3.64}$$

Inserting these quantities into both sides of the Cartan-Evans identity, we find that both sides are equal, thus the identity holds in the chosen example. ∎

3.3 Alternative forms of Cartan-Bianchi and Cartan-Evans identity

3.3.1 Cartan-Evans identity

We showed that the Cartan-Evans identity is based on the fundamental definition of the Hodge dual torsion and curvature, and adds three of them in cyclic permutation.

By using the definition

$$\widetilde{T}^{a}{}_{\mu\nu} = q^{a}{}_{\lambda} \widetilde{T}^{\lambda}{}_{\mu\nu} \tag{3.65}$$

it follows that:

$$D_\mu \widetilde{T}^{a}{}_{\nu\rho} = (D_\mu q^{a}{}_{\kappa}) \widetilde{T}^{\kappa}{}_{\nu\rho} + q^{a}{}_{\kappa} D_\mu \widetilde{T}^{\kappa}{}_{\nu\rho} \tag{3.66}$$

using the Leibniz rule. We use the tetrad postulate:

$$D_\mu q^{a}{}_{\kappa} = 0 \tag{3.67}$$

to find that:

$$D_\mu \widetilde{T}^a{}_{\nu\rho} = q^a{}_\kappa D_\mu \widetilde{T}^\kappa{}_{\nu\rho} \,. \tag{3.68}$$

It follows that:

$$D_\mu \widetilde{T}^\kappa{}_{\nu\rho} + D_\nu \widetilde{T}^\kappa{}_{\rho\mu} + D_\rho \widetilde{T}^\kappa{}_{\mu\nu} = \widetilde{R}^\kappa{}_{\mu\nu\rho} + \widetilde{R}^\kappa{}_{\nu\rho\mu} + \widetilde{R}^\kappa{}_{\rho\mu\nu} \tag{3.69}$$

which is the Cartan-Evans identity written in the base manifold only. This equation may be rewritten as:

$$D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}. \tag{3.70}$$

The easiest way to see this is to take a particular example:

$$D_1 \widetilde{T}^\kappa{}_{23} + D_3 \widetilde{T}^\kappa{}_{12} + D_2 \widetilde{T}^\kappa{}_{31} = \widetilde{R}^\kappa{}_{123} + \widetilde{R}^\kappa{}_{312} + \widetilde{R}^\kappa{}_{231} \tag{3.71}$$

and then to take Hodge dual terms with upper indices according to Eq. (2.117). The constant factors cancel out. For the Levi-Civita symbol, the relation holds:

$$\varepsilon^{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}, \tag{3.72}$$

so that the sign change also cancels out. Furthermore, for a two-fold Hodge dual of a tensor T, the relation

$$\widetilde{\widetilde{T}} = \pm T \tag{3.73}$$

is valid, so that any sign change of this kind also cancels out. We take the Hodge dual of (3.71) term by term. The Levi-Civita symbol effects that in the expressions

$$\varepsilon^{\mu\nu\alpha\beta} T^\kappa{}_{\alpha\beta} \tag{3.74}$$

the index pairs $(\mu\nu)$ and $(\alpha\beta)$ are mutually exclusive:

$$\begin{aligned} &\mu \neq \nu, \quad \alpha \neq \beta, \\ &\mu \notin \{\alpha, \beta\}, \\ &\nu \notin \{\alpha, \beta\}. \end{aligned} \tag{3.75}$$

In total, we obtain for the Hodge dual example of (3.71):

$$D_1 T^{\kappa 01} + D_2 T^{\kappa 02} + D_3 T^{\kappa 03} = R^\kappa{}_1{}^{01} + R^\kappa{}_2{}^{02} + R^\kappa{}_3{}^{03} \tag{3.76}$$

which is an example of Eq. (3.70), the alternative form of the Cartan-Evans identity:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}} \,. \tag{3.77}$$

3.3.2 Cartan-Bianchi identity

Eq. (3.77) is the most useful format of the Cartan-Evans identity. The Cartan-Bianchi identity can also be rewritten into this format. From Eq. (3.19) follows:

$$D_\mu T^\kappa{}_{\nu\rho} + D_\nu T^\kappa{}_{\rho\mu} + D_\rho T^\kappa{}_{\mu\nu} = R^\kappa{}_{\mu\nu\rho} + R^\kappa{}_{\nu\rho\mu} + R^\kappa{}_{\rho\mu\nu} \tag{3.78}$$

which is identical to (3.69), except that these are the original tensors instead of the Hodge duals. Therefore, the same derivation as above leads to the alternative form of the Cartan-Bianchi-identity:

$$\boxed{D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu}} \,. \tag{3.79}$$

It should be noted that in the above contravariant forms of both identities, the Hodge dual and original tensors are interchanged, compared to the covariant forms (3.19) and (3.45).

3.3.3 Consequences of the identities

At the end of this section we will investigate the implications of antisymmetry of the Gamma connection in Cartan geometry. The Gamma connection has to have at least antisymmetric parts with

$$\Gamma^{\lambda}{}_{\mu\nu} = -\Gamma^{\lambda}{}_{\nu\mu}. \tag{3.80}$$

If $\mu = \nu$, the commutator vanishes, as do the torsion and curvature tensors. If there are only symmetric parts in the connection:

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \neq ? \, 0 \tag{3.81}$$

then torsion vanishes, leading to the special case of (3.77):

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} = 0. \tag{3.82}$$

It has been shown by computer algebra [16, 17] that all of the metrics of the Einstein field equation in the presence of matter give the erroneous result:

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq ? \, 0, \tag{3.83}$$

$$D_{\mu} T^{\kappa\mu\nu} = ? \, 0. \tag{3.84}$$

This contradicts basic properties of Cartan geometry, the superset of Riemannian geometry, and therefore Eq. (3.77) is a constraint for theories like Einsteinian relativity, which are based on Riemannian geometry. This error has been perpetuated uncritically for nearly a hundred years, and allowed to create a defective cosmology that should be discarded by scholars. The cosmology of the Standard Model is baseless and incorrect, and should be replaced by ECE cosmology, which is based on torsion.

3.4 Further identities

There are some other identities which are not as significant to the field equations of ECE theory, but which represent new insights into Cartan geometry. They were developed as part of ECE theory, and we present them here, partially without proofs (which can be found in the Unified Field Theory (UFT) Section of www.aias.us).

3.4.1 Evans torsion identity (first Evans identity)

From the Cartan-Bianchi identity another identity can be derived, containing torsion terms only. This is the *Evans torsion identity* [14]. In explicit form it reads

$$T^{\kappa}{}_{\lambda\nu} T^{\lambda}{}_{\sigma\mu} + T^{\kappa}{}_{\lambda\mu} T^{\lambda}{}_{\nu\sigma} + T^{\kappa}{}_{\lambda\sigma} T^{\lambda}{}_{\mu\nu} = 0 \tag{3.85}$$

and can be written in short form with permutation brackets:

$$\boxed{T^{\kappa}{}_{\lambda[\nu} T^{\lambda}{}_{\sigma\mu]} = 0}. \tag{3.86}$$

The identity can be rewritten in form notation as

$$T^{\kappa}{}_{\lambda} \wedge T^{\lambda} = 0, \tag{3.87}$$

or by multiplying with $q^{a}{}_{\kappa}$:

$$\boxed{T^{a}{}_{\lambda} \wedge T^{\lambda} = 0}. \tag{3.88}$$

Here $T^{a}{}_{\lambda}$ is a 1-form and T^{λ} is a 2-form, making up a 3-form on the left-hand side. The proof consists mainly of inserting the definitions of torsion into the Cartan-Bianchi identity, and can be found in the literature [14].

3.4.2 Jacobi identity

The Jacobi identity [18] is an exact identity used in field theory and general relativity. It is an operator identity that applies to covariant derivatives and group generators [19] alike. It is very rarely proven in all detail, so we are providing the following complete proof. The *Jacobi identity* is a permuted sum of three covariant derivatives:

$$\boxed{[D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]=0}\,. \tag{3.89}$$

For the proof, we expand the commutators on the-left hand side:

$$\begin{aligned} L.H.S =&[D_\rho,D_\mu D_\nu - D_\nu D_\mu]+[D_\nu,D_\rho D_\mu - D_\mu,D_\rho]+[D_\mu,D_\nu D_\rho - D_\rho D_\nu] \\ =&D_\rho(D_\mu D_\nu - D_\nu D_\mu)-(D_\mu D_\nu - D_\nu D_\mu)D_\rho \\ &+D_\nu(D_\rho D_\mu - D_\mu,D_\rho)-(D_\rho D_\mu - D_\mu,D_\rho)D_\nu \\ &+D_\mu(D_\nu D_\rho - D_\rho D_\nu)-(D_\nu D_\rho - D_\rho D_\nu)D_\mu, \end{aligned} \tag{3.90}$$

and this expansion is regarded as an expansion by algebra which sums up to zero:

$$\begin{aligned} L.H.S =&D_\rho D_\mu D_\nu - D_\rho D_\nu D_\mu - D_\mu D_\nu D_\rho + D_\nu D_\mu D_\rho \\ &+D_\nu D_\rho D_\mu - D_\nu D_\mu D_\rho - D_\rho D_\mu D_\nu + D_\mu D_\rho D_\nu \\ &+D_\mu D_\nu D_\rho - D_\mu D_\rho D_\nu - D_\nu D_\rho D_\mu + D_\rho D_\nu D_\mu \\ =&0, \end{aligned} \tag{3.91}$$

Q.E.D. The Jacobi identity can also be written in an alternative form:

$$\boxed{[[D_\mu,D_\nu],D_\rho]+[[D_\rho,D_\mu],D_\nu]+[[D_\nu,D_\rho],D_\mu]=0}\,. \tag{3.92}$$

3.4.3 Bianchi-Cartan-Evans identity

Einsteinian general relativity uses the second Bianchi identity, which is obtained from the covariant derivative of the first Bianchi identity. General relativity ignores torsion, but the same procedure can be applied to the Cartan-Bianchi identity of Cartan geometry, which contains both torsion and curvature. The result is the *Bianchi-Cartan-Evans identity* [20, 21, 22]:

$$\boxed{D_\mu D_\lambda T^\kappa{}_{\nu\rho} + D_\rho D_\lambda T^\kappa{}_{\mu\nu} + D_\nu D_\lambda T^\kappa{}_{\rho\mu} = D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu}}\,. \tag{3.93}$$

The proof was first carried out in UFT Paper 88 [20] which is the most read paper of ECE theory. Two variants of this identity can be produced by cyclic permutation of (μ,ν,ρ). Eq. (3.93) is the correct "second Bianchi identity" augmented by torsion. In Einsteinian theory the incorrect version

$$D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} =? 0 \tag{3.94}$$

is used. This follows from (3.93) by arbitrarily omitting torsion. The Einstein-Hilbert field equation is derived from this erroneously truncated "second Bianchi identity" [20]. Therefore, all solutions of the Einstein-Hilbert field equation are inconsistent.

It should be noted that the Bianchi-Cartan-Evans identity gives no information beyond what is provided by the Cartan-Bianchi identity, because it is derived from the latter by differentiation and therefore not independent.

3.4.4 Jacobi-Cartan-Evans identity

The Jacobi identity can be used to derive another identity. When the terms of the Cartan-Bianchi identity are inserted into the Jacobi identity (3.89), the relation

$$\begin{aligned}&\left([D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]\right)V^\kappa \\ &=\left(D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \\ &\quad-\left(T^\lambda{}_{\mu\nu}\,[D_\rho,D_\lambda]+T^\lambda{}_{\rho\mu}\,[D_\nu,D_\lambda]+T^\lambda{}_{\nu\rho}\,[D_\mu,D_\lambda]\right)V^\kappa \\ &=0\end{aligned} \tag{3.95}$$

follows, where the Jacobi identity has been applied to an arbitrary vector V^κ of the base manifold. Because the Jacobi identity sums to zero, we obtain the equation

$$\begin{aligned}&\left(D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \\ &=\left(T^\lambda{}_{\mu\nu}\,[D_\rho,D_\lambda]+T^\lambda{}_{\rho\mu}\,[D_\nu,D_\lambda]+T^\lambda{}_{\nu\rho}\,[D_\mu,D_\lambda]\right)V^\kappa.\end{aligned} \tag{3.96}$$

Further transformations, described in [22], give

$$\boxed{D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}=T^\alpha{}_{\mu\nu}\,R^\kappa{}_{\lambda\rho\alpha}+T^\alpha{}_{\rho\mu}\,R^\kappa{}_{\lambda\nu\alpha}+T^\alpha{}_{\nu\rho}\,R^\kappa{}_{\lambda\mu\alpha}}, \tag{3.97}$$

which is called the *Jacobi-Cartan-Evans identity*.

Part Two: Electrodynamics

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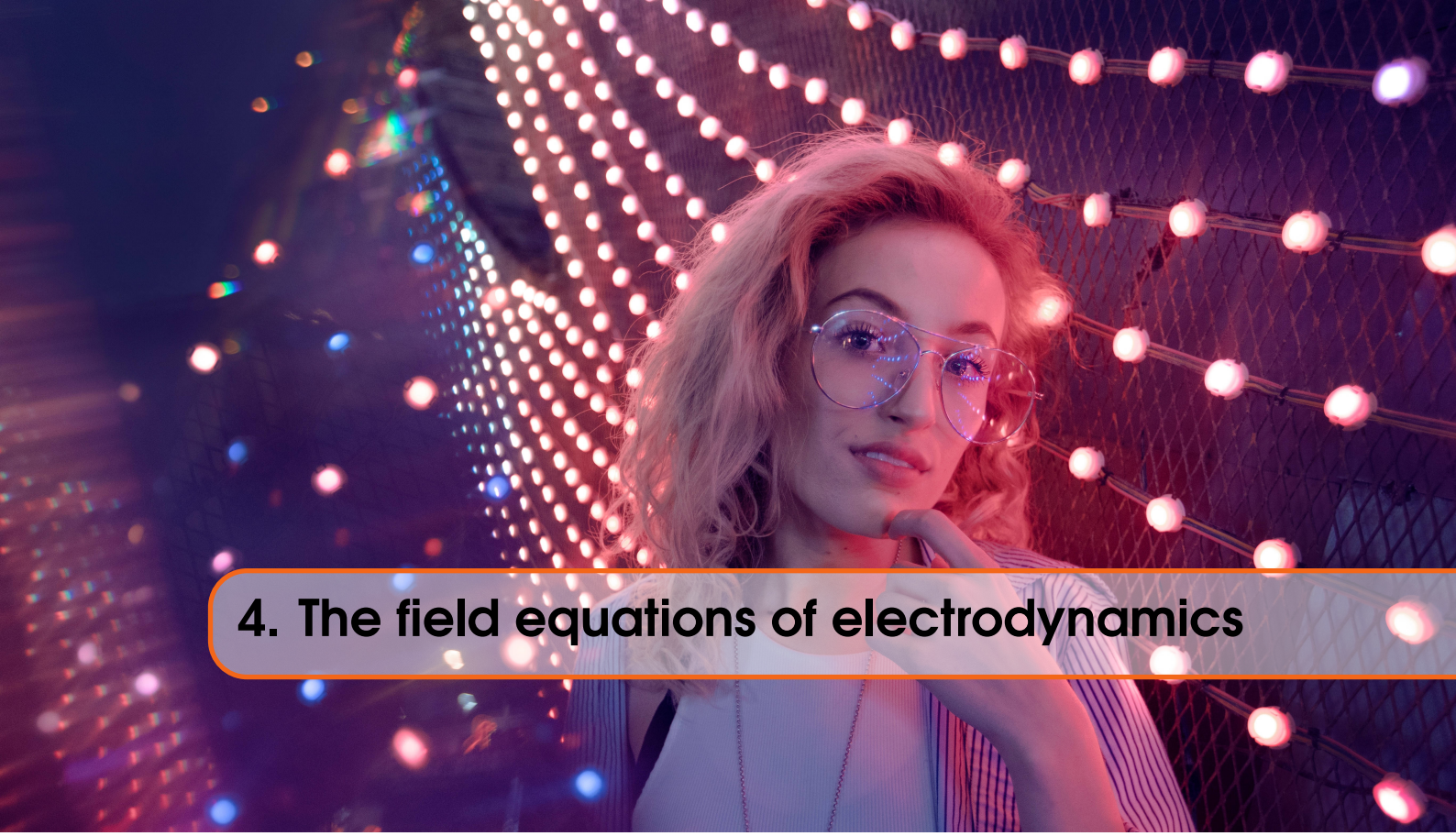

4. The field equations of electrodynamics

In the preceding chapter, we developed the mathematical methodology for ECE theory: Cartan geometry and its most important theorems. Now we switch our focus to physics. We first describe how physical quantities are obtained from geometrical quantities. This is achieved by defining suitable axioms. We then derive the field equations of electromagnetism, as well as the wave equation. In this way, electrodynamics is transformed into an axiomatic, mathematically correct theory. It is shown how the spin connections extend classical electromagnetism to a theory of general relativity. The novel concepts are underpinned by a number of important applications.

4.1 The axioms

In order to obtain physical quantities, we have first to define how geometry is transformed into physics. We do this by two fundamental axioms, relating to potentials and electromagnetic fields. The first axiom states that the electromagnetic potential is proportional to the Cartan tetrad. Thus, the geometry of spacetime is directly equated to physical quantities. The potential contains the same indices as the tetrad. We use the 4-vector potential A_μ in relativistic notation. However, the tetrad $q^a{}_\mu$ is (formally) a matrix and contains the polarization index a. Therefore, the potential is extended to matrix form with two indices: $A^a{}_\mu$. This is the main formal difference from classical electrodynamics: all electromagnetic ECE quantities have a polarization index, extending the definition range by one dimension. We will see later how we can reduce these quantities to one polarization direction, if required.

The first axiom is formally written in the form

$$A^a{}_\mu := A^{(0)} q^a{}_\mu, \tag{4.1}$$

where we have introduced a factor of proportionality $A^{(0)}$. Since the tetrad is dimensionless, $A^{(0)}$ must have the physical units of a vector potential which is Vs/m or $T \cdot m$. The product $c \cdot A^{(0)}$ can be considered as a primordial voltage, where c is the velocity of light. In detailed form, the complete

ECE potential reads:

$$A^a{}_\mu = \begin{bmatrix} A^{(0)}{}_0 & A^{(0)}{}_1 & A^{(0)}{}_2 & A^{(0)}{}_3 \\ A^{(1)}{}_0 & A^{(1)}{}_1 & A^{(1)}{}_2 & A^{(1)}{}_3 \\ A^{(2)}{}_0 & A^{(2)}{}_1 & A^{(2)}{}_2 & A^{(2)}{}_3 \\ A^{(3)}{}_0 & A^{(3)}{}_1 & A^{(3)}{}_2 & A^{(3)}{}_3 \end{bmatrix} = A^{(0)} \begin{bmatrix} q^{(0)}{}_0 & q^{(0)}{}_1 & q^{(0)}{}_2 & q^{(0)}{}_3 \\ q^{(1)}{}_0 & q^{(1)}{}_1 & q^{(1)}{}_2 & q^{(1)}{}_3 \\ q^{(2)}{}_0 & q^{(2)}{}_1 & q^{(2)}{}_2 & q^{(2)}{}_3 \\ q^{(3)}{}_0 & q^{(3)}{}_1 & q^{(3)}{}_2 & q^{(3)}{}_3 \end{bmatrix}. \tag{4.2}$$

The lines number the polarization and the columns the coordinate indices. The zeroth component of the potential is the scalar potential ϕ, which also gets a polarization index:

$$A^a{}_0 = \frac{\phi^a}{c}. \tag{4.3}$$

The above equations use the mixed-index notation (contravariant and covariant). The coordinates of vectors, however, correspond to contravariant indices. Therefore, we transform the coordinate indices by the Minkowski metric, Eq. (2.39):

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.4}$$

in the usual form

$$A^{a\mu} = \eta^{\mu\nu} A^a{}_\nu. \tag{4.5}$$

The diagonal form of the Minkowski metric leads to

$$\begin{aligned} A^{(0)0} &= \eta^{00} A^{(0)}{}_0 = A^{(0)}{}_0, \\ A^{(0)1} &= \eta^{11} A^{(0)}{}_1 = -A^{(0)}{}_1, \\ A^{(0)2} &= \eta^{22} A^{(0)}{}_2 = -A^{(0)}{}_2, \\ &\text{etc.} \end{aligned} \tag{4.6}$$

so that the potential components with coordinates 1,2,3 are changed in sign:

$$A^{a\mu} = \begin{bmatrix} A^{(0)0} & A^{(0)1} & A^{(0)2} & A^{(0)3} \\ A^{(1)0} & A^{(1)1} & A^{(1)2} & A^{(1)3} \\ A^{(2)0} & A^{(2)1} & A^{(2)2} & A^{(2)3} \\ A^{(3)0} & A^{(3)1} & A^{(3)2} & A^{(3)3} \end{bmatrix} = \begin{bmatrix} A^{(0)}{}_0 & A^{(0)}{}_1 & A^{(0)}{}_2 & A^{(0)}{}_3 \\ A^{(1)}{}_0 & -A^{(1)}{}_1 & -A^{(1)}{}_2 & -A^{(1)}{}_3 \\ A^{(2)}{}_0 & -A^{(2)}{}_1 & -A^{(2)}{}_2 & -A^{(2)}{}_3 \\ A^{(3)}{}_0 & -A^{(3)}{}_1 & -A^{(3)}{}_2 & -A^{(3)}{}_3 \end{bmatrix}. \tag{4.7}$$

This is the form of the potential we will mostly use.

The relativistic electromagnetic field tensor $F^{\mu\nu}$ was already introduced in Example 2.11 and its Hodge dual in Example 2.12. These are 2-index tensors, comprising the electric and magnetic field. Therefore, the electromagnetic field of ECE theory also has to have two coordinate indices. We define the electromagnetic field tensor of ECE theory to be proportional to the Cartan torsion $T^a{}_{\mu\nu}$:

$$F^a{}_{\mu\nu} := A^{(0)} T^a{}_{\mu\nu}. \tag{4.8}$$

Because torsion has a polarization index, the electromagnetic field has to have one, too. It is a 3-index quantity, an indexed antisymmetric 2-form of Cartan geometry. In classical electromagnetism, the electromagnetic field is a derivative of the potential, therefore it has the units of

[potential]/[length] = $Vs/m^2 = T$. Since geometrical torsion has the units $1/m$, the constant of proportionality $A^{(0)}$ is the same for both the potential and the field. The polarization index can be seen as a vector index augmenting the electric field $\mathbf{E}$ and magnetic field (i.e. induction) $\mathbf{B}$. Therefore, we have fields $\mathbf{E}^a$ and $\mathbf{B}^a$, which are components of the ECE electromagnetic tensor field:

$$F^{a\mu\nu} = \begin{bmatrix} F^{a00} & F^{a01} & F^{a02} & F^{a03} \\ F^{a10} & F^{a11} & F^{a12} & F^{a13} \\ F^{a20} & F^{a21} & F^{a22} & F^{a23} \\ F^{a30} & F^{a31} & F^{a32} & F^{a33} \end{bmatrix} = \begin{bmatrix} 0 & -E^{a1} & -E^{a2} & -E^{a3} \\ E^{a1} & 0 & -cB^{a3} & cB^{a2} \\ E^{a2} & cB^{a3} & 0 & -cB^{a1} \\ E^{a3} & -cB^{a2} & cB^{a1} & 0 \end{bmatrix}. \tag{4.9}$$

The Hodge dual of the classical electromagnetic field $F^{\mu\nu}$ was computed in Example 2.12. As demonstrated before, this has to be augmented by a polarization or tangent space index a, leading to

$$\widetilde{F}^{a\mu\nu} = \begin{bmatrix} 0 & cB^{a1} & cB^{a2} & cB^{a3} \\ -cB^{a1} & 0 & -E^{a3} & E^{a2} \\ -cB^{a2} & E^{a3} & 0 & -E^{a1} \\ -cB^{a3} & -E^{a2} & E^{a1} & 0 \end{bmatrix}. \tag{4.10}$$

We have chosen the contravariant versions of F and $\widetilde{F}$, because these correspond to the electric and magnetic vector components and are needed for deriving the field equations in vector form.

The ECE potential is a Cartan 1-form and the ECE electromagnetic field is a Cartan 2-form. Both are vector-valued by the polarization index a. In summary, the basic ECE axioms are:

$$A^a{}_\mu := A^{(0)} q^a{}_\mu, \tag{4.11}$$

$$F^a_{\mu\nu} := A^{(0)} T^a_{\mu\nu}. \tag{4.12}$$

4.2 The field equations

The aim of this section is to derive the ECE field equations in the form of Maxwell's equations. The latter are known as:

Gauss' law:

$$\nabla \cdot \mathbf{B} = 0, \tag{4.13}$$

Faraday's law of induction:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{4.14}$$

Coulomb's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{4.15}$$

Ampère-Maxwell's law:

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \tag{4.16}$$

where ρ is the electrical charge density and $\mathbf{J}$ the current density. The ECE field equations will be shown to be identical to the Cartan-Bianchi identity and the Cartan-Evans identity. These theorems of geometry are converted into physical laws by multiplying them with the factor $A^{(0)}$.

4.2.1 The field equations in covariant tensor form

From Eqs. (3.20) and (3.46), in form notation,

$$D \wedge T^a = R^a{}_b \wedge q^b, \tag{4.17}$$
$$D \wedge \widetilde{T}^a = \widetilde{R}^a{}_b \wedge q^b, \tag{4.18}$$

follows, when inserting the ECE Axioms (4.11) and (4.12):

$$D \wedge F^a = R^a{}_b \wedge A^b, \tag{4.19}$$
$$D \wedge \widetilde{F}^a = \widetilde{R}^a{}_b \wedge A^b. \tag{4.20}$$

These are the field equations, written in 3-forms at both sides of the equations. We will see that these two equations lead to the equivalent of Maxwell's equations (4.13 - 4.16). To develop this, we rewrite the field equations in tensor form first. With indices written out, they read

$$(D \wedge F^a)_{\mu\nu\rho} = (R^a{}_b \wedge A^b)_{\mu\nu\rho}, \tag{4.21}$$
$$(D \wedge \widetilde{F}^a)_{\mu\nu\rho} = (\widetilde{R}^a{}_b \wedge A^b)_{\mu\nu\rho}, \tag{4.22}$$

or, with wedge operation carried out, in tensor form:

$$D_{[\mu} F^a{}_{\nu\rho]} = R^a{}_{b[\mu\nu} A^b{}_{\rho]}, \tag{4.23}$$
$$D_{[\mu} \widetilde{F}^a{}_{\nu\rho]} = \widetilde{R}^a{}_{b[\mu\nu} A^b{}_{\rho]}. \tag{4.24}$$

The covariant exterior derivative $D\wedge$ was defined in Eqs. (3.2/3.3):

$$(D \wedge F^a)_{\mu\nu\rho} = (d \wedge F^a)_{\mu\nu\rho} + (\omega^a{}_b \wedge F^b)_{\mu\nu\rho}, \tag{4.25}$$

or, written as a cyclic sum:

$$D_{[\mu} F^a{}_{\nu\rho]} = \partial_{[\mu} F^a{}_{\nu\rho]} + \omega^a{}_{[\mu b} F^b{}_{\nu\rho]}\,. \tag{4.26}$$

Inserting this derivative into the first field equation (4.23) gives (in form notation):

$$d \wedge F^a + \omega^a{}_b \wedge F^b = R^a{}_b \wedge A^b, \tag{4.27}$$

and bringing the spin connection term to the right-hand side gives:

$$d \wedge F^a = R^a{}_b \wedge A^b - \omega^a{}_b \wedge F^b. \tag{4.28}$$

This equation has a form similar to the first two field equations of classical electrodynamics, Eq. (2.173) in Example 2.11, where they are condensed into one equation:

$$d \wedge F = 0. \tag{4.29}$$

These two field equations (the Gauss law and the Faraday law) are homogeneous, i.e. there are no current terms on the right-hand side. In contrast, the right-hand side of (4.28) is not zero. It is therefore to be assumed that a current may exist for the Gauss and Faraday laws in their generalized form in ECE theory. This is called the *homogeneous current* and denoted by j. It corresponds to magnetic charges and currents, whose existence is not universally accepted in science.

In ECE theory, the homogeneous current has to be augmented by a polarization index so that Eq. (4.28) can be written as

$$d \wedge F^a = j^a \tag{4.30}$$

with the definition of the homogeneous current being

$$j^a := R^a{}_b \wedge A^b - \omega^a{}_b \wedge F^b. \tag{4.31}$$

For the second field equation (4.24), containing the Hodge duals $\widetilde{F}$ and $\widetilde{R}$, we have to use the spin connection of the Λ connection as defined by Eq. (3.49). For clarity, we add an index (Λ) here:

$$d \wedge \widetilde{F}^a = \widetilde{R}^a{}_b \wedge A^b - \omega_{(\Lambda)}{}^a{}_b \wedge \widetilde{F}^b. \tag{4.32}$$

This equation defines the other pair of generalized Maxwell equations, the Coulomb and Ampère-Maxwell laws. We find

$$d \wedge \widetilde{F}^a = \mu_0 J^a \tag{4.33}$$

with the definition

$$J^a := \frac{1}{\mu_0}\left(\widetilde{R}^a{}_b \wedge A^b - \omega_{(\Lambda)}{}^a{}_b \wedge \widetilde{F}^b\right), \tag{4.34}$$

which we call the *inhomogeneous current*. This corresponds to the well-known electrical 4-current density. In addition, we have introduced the vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{Vs}}{\mathrm{Am}}$ in order to obtain J^a in the usual units of $\mathrm{A/m}^2$. The 0-component of J^a is the electric charge density ρ, augmented by a polarization index. For consistency, we will also use the factor $1/\mu_0$ in the homogeneous current (4.31).

Please note that the current densities here are 3-forms. For example, in tensor notation, Eq. (4.34) reads:

$$\begin{aligned}(J^a)_{\mu\nu\rho} &= \frac{1}{\mu_0}\left(R^a{}_{b[\mu\nu}A^b{}_{\rho]} - \omega_{(\Lambda)}{}^a{}_{[\mu b}F^b{}_{\nu\rho]}\right). \\ &= \frac{1}{\mu_0}\Big(R^a{}_{b\mu\nu}A^b{}_\rho + R^a{}_{b\nu\rho}A^b{}_\mu + R^a{}_{b\rho\mu}A^b{}_\nu \\ &\quad - \omega_{(\Lambda)}{}^a{}_{\mu b}F^b{}_{\nu\rho} - \omega_{(\Lambda)}{}^a{}_{\nu b}F^b{}_{\rho\mu} - \omega_{(\Lambda)}{}^a{}_{\rho b}F^b{}_{\mu\nu}\Big).\end{aligned} \tag{4.35}$$

The standard current density, however, is a 1-form, because the current density $\mathbf{J}$ is a vector with only one coordinate index. So, j^a and J^a are types of generalized currents, which cannot simply be correlated to known quantities. However, we will see in the next section that the contravariant formulation reduces them to 1-forms, as requested.

So far, we can write the ECE field equations in form notation with covariant tensors:

$$\boxed{\begin{aligned} d \wedge F^a &= \mu_0\, j^a, \\ d \wedge \widetilde{F}^a &= \mu_0\, J^a. \end{aligned}} \tag{4.36, 4.37}$$

For formal symmetry, we have added a factor of μ_0 here to the homogeneous current j^a. This is arbitrary and only changes the units of j^a, which of course are different from those of J^a.

4.2.2 The field equations in contravariant tensor form

In this section, the field equations are translaated into vector form so that they will be familiar to engineers and physicists. We start with the alternative form of the Cartan-Bianchi and Cartan-Evans identities, Eqs. (3.79 and 3.77).

$$D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu}, \tag{4.38}$$

$$D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}. \tag{4.39}$$

These equations were given in the base manifold, but can be transformed to tangent space. The κ index is replaced formally by the a index, multiplying the equations with $q^a{}_\kappa$ and applying the tetrad postulate:

$$D_\mu \widetilde{T}^{a\mu\nu} = \widetilde{R}^a{}_\mu{}^{\mu\nu}, \tag{4.40}$$
$$D_\mu T^{a\mu\nu} = R^a{}_\mu{}^{\mu\nu}. \tag{4.41}$$

Multiplying by the factor $A^{(0)}$ then gives us the second form of field equations:

$$D_\mu \widetilde{F}^{a\mu\nu} = A^{(0)} \widetilde{R}^a{}_\mu{}^{\mu\nu}, \tag{4.42}$$
$$D_\mu F^{a\mu\nu} = A^{(0)} R^a{}_\mu{}^{\mu\nu}. \tag{4.43}$$

Please notice that the role of the original and Hodge dual equations have interchanged, compared to (4.36) and (4.37). The first equation, corresponding to the first pair of Maxwell equations, is based on the Hodge dual equation, and the second equation, corresponding to the third and forth Maxwell equations, contains the original torsion and curvature. There is no wedge product in the equations but there is a summation over the coordinate parameter μ. They are tensor equations with a contraction.

Now we apply the definition of the covariant derivative, similarly as in the preceding section:

$$D_\mu F^{a\nu\rho} = \partial_\mu F^{a\nu\rho} + \omega^a{}_{\mu b} F^{b\nu\rho}, \tag{4.44}$$

leading to

$$\partial_\mu \widetilde{F}^{a\mu\nu} = A^{(0)} \widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega_{(\Lambda)}{}^a{}_{\mu b} \widetilde{F}^{b\mu\nu}, \tag{4.45}$$
$$\partial_\mu F^{a\mu\nu} = A^{(0)} R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} F^{b\mu\nu}. \tag{4.46}$$

The first equation is similar to Eq. (2.185), representing the first two Maxwell equations of classical electrodynamics in Hodge dual formulation. Therefore, we can again interpret the right-hand sides of both equations as an homogeneous and inhomogeneous current. Similarly to (4.36, 4.37) we can write:

$$\boxed{\begin{aligned} \partial_\mu \widetilde{F}^{a\mu\nu} &= \mu_0 j^{a\nu}, \\ \partial_\mu F^{a\mu\nu} &= \mu_0 J^{a\nu} \end{aligned}} \tag{4.47, 4.48}$$

with

$$j^{a\nu} := \frac{1}{\mu_0}\left(A^{(0)} \widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega_{(\Lambda)}{}^a{}_{\mu b} \widetilde{F}^{b\mu\nu}\right), \tag{4.49}$$
$$J^{a\nu} := \frac{1}{\mu_0}\left(A^{(0)} R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} F^{b\mu\nu}\right). \tag{4.50}$$

Now we have arrived at 1-forms for the currents, which can be set into relation with classical expressions. The covariant 4-current density is written in tangent space as

$$(J^a)_\mu = \begin{bmatrix} J^a{}_0 \\ J^a{}_1 \\ J^a{}_2 \\ J^a{}_3 \end{bmatrix}. \tag{4.51}$$

To use the components in the usual contravariant form we have to raise the coordinate indices, which gives a sign change of space components according to the Minkowski metric:

$$(J^a)^\nu = \begin{bmatrix} J^{a0} \\ J^{a1} \\ J^{a2} \\ J^{a3} \end{bmatrix} = \begin{bmatrix} J^a{}_0 \\ -J^a{}_1 \\ -J^a{}_2 \\ -J^a{}_3 \end{bmatrix}. \tag{4.52}$$

The 0-component is defined by

$$J^{a0} = c\rho^a. \tag{4.53}$$

The right-hand side of J^{a0} has the units of $\frac{m}{s}\frac{C}{m^3} = \frac{C}{m^2 s} = \frac{A}{m^2}$, which is the same current density unit as used for the spatial components.

We conclude this section with the hint that the currents are geometrical quantities and not externally imposed as in Maxwell's theory. The field equations are fully geometric, no terms are added defining an external energy-momentum like in Einstein's general relativity. Since the currents depends on the fields F, for which the equations have to be solved, we have an intrinsic nonlinearity. A similar case is known from Ohm's law where the current density is assumed to be proportional to the electric field via the conductivity σ, which in general is a tensor. In most cases, σ is assumed to be a scalar quantity, and Ohm's law is used for the current term in classical electrodynamics:

$$\mathbf{J} = \sigma \mathbf{E}. \tag{4.54}$$

Comparing this with Eq. (4.50), the conductivity takes the role of a constant scalar spin connection. However, the ECE field equations are valid in a curved and twisted spacetime, going far beyond the Minkowski space of Maxwell's equations and special relativity.

4.2.3 The field equations in vector form

It was already demonstrated in Example 2.12, how the Gauss and Faraday laws are derived from a tensor equation of the Hodge dual of the classical electromagnetic field, $\widetilde{F}^{\mu\nu}$. It is easy to extend this procedure to the field equation (4.47):

$$\partial_\mu \widetilde{F}^{a\mu\nu} = \mu_0 j^{a\nu}. \tag{4.55}$$

According to Eq. (4.10), the field is an antisymmetric tensor, consisting of electric and magnetic field components. In Examples 2.11 and 2.12, the field tensor was given in electric field units of V/m for convenience. We have the freedom of choice for these units. Here we use the units of the magnetic field (Tesla) so that we obtain the same constants in the vector equations aswe do in Maxwell's equations. This means that we have to make the following replacements:

$$E^\mu \to E^\mu / c,$$
$$cB^\mu \to B^\mu.$$

In addition, the fields have to be augmented by the polarization index a:

$$\widetilde{F}^{a\mu\nu} = \begin{bmatrix} \widetilde{F}^{a00} & \widetilde{F}^{a01} & \widetilde{F}^{a02} & \widetilde{F}^{a03} \\ \widetilde{F}^{a10} & \widetilde{F}^{a11} & \widetilde{F}^{a12} & \widetilde{F}^{a13} \\ \widetilde{F}^{a20} & \widetilde{F}^{a21} & \widetilde{F}^{a22} & \widetilde{F}^{a23} \\ \widetilde{F}^{a30} & \widetilde{F}^{a31} & \widetilde{F}^{a32} & \widetilde{F}^{a33} \end{bmatrix} = \begin{bmatrix} 0 & B^{a1} & B^{a2} & B^{a3} \\ -B^{a1} & 0 & -E^{a3}/c & E^{a2}/c \\ -B^{a2} & E^{a3}/c & 0 & -E^{a1}/c \\ -B^{a3} & -E^{a2}/c & E^{a1}/c & 0 \end{bmatrix}. \tag{4.56}$$

The homogeneous field equations are obtained by specific selection of indices, in the following way. Eq. (4.55) consists of four equations ($\nu = 0, \ldots, 3$), each with four summands of μ on the

left-hand side. The case $\mu = \nu$ leads to diagonal elements of F, so these terms can be omitted. The generalized Gauss law is obtained by choosing $\nu = 0$, which leads to

$$\partial_1 \widetilde{F}^{a10} + \partial_2 \widetilde{F}^{a20} + \partial_3 \widetilde{F}^{a30} = -\partial_1 B^{a1} - \partial_2 B^{a2} - \partial_3 B^{a3} = \mu_0 j^{a0}. \tag{4.57}$$

In vector notation this is

$$\nabla \cdot \mathbf{B}^a = -\mu_0 j^{a0}. \tag{4.58}$$

The Faraday law of induction is obtained by choosing $\nu = 1,2,3$ and consists of three component equations:

$$\begin{aligned} \partial_0 \widetilde{F}^{a01} + \partial_2 \widetilde{F}^{a21} + \partial_3 \widetilde{F}^{a31} &= \mu_0 j^{a1}, \\ \partial_0 \widetilde{F}^{a02} + \partial_1 \widetilde{F}^{a12} + \partial_3 \widetilde{F}^{a32} &= \mu_0 j^{a2}, \\ \partial_0 \widetilde{F}^{a03} + \partial_1 \widetilde{F}^{a13} + \partial_2 \widetilde{F}^{a23} &= \mu_0 j^{a3}. \end{aligned} \tag{4.59}$$

These can be written, according to (4.56), as

$$\begin{aligned} \partial_0 B^{a1} + \partial_2 E^{a3}/c - \partial_3 E^{a2}/c &= \mu_0 j^{a1}, \\ \partial_0 B^{a2} - \partial_1 E^{a3}/c + \partial_3 E^{a1}/c &= \mu_0 j^{a2}, \\ \partial_0 B^{a3} + \partial_1 E^{a2}/c - \partial_2 E^{a1}/c &= \mu_0 j^{a3}. \end{aligned} \tag{4.60}$$

Taking into account $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$, these equations can be condensed into one vector equation, which is

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = c\,\mu_0 \mathbf{j}^a. \tag{4.61}$$

We see that these "homogeneous" equations are not actually homogeneous, because there is a magnetic charge density j^{a0} and a magnetic current vector $\mathbf{j}^a$, in general. In nearly all practical applications, however, we will set

$$j^{a0} = 0, \tag{4.62}$$

$$\mathbf{j}^a = \mathbf{0}. \tag{4.63}$$

The Coulomb and Ampère-Maxwell laws are derived in a completely analogous way from Eq. (4.48):

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu}. \tag{4.64}$$

According to Eq. (4.9), the contravariant field tensor (in Tesla units) is:

$$F^{a\mu\nu} = \begin{bmatrix} F^{a00} & F^{a01} & F^{a02} & F^{a03} \\ F^{a10} & F^{a11} & F^{a12} & F^{a13} \\ F^{a20} & F^{a21} & F^{a22} & F^{a23} \\ F^{a30} & F^{a31} & F^{a32} & F^{a33} \end{bmatrix} = \begin{bmatrix} 0 & -E^{a1}/c & -E^{a2}/c & -E^{a3}/c \\ E^{a1}/c & 0 & -B^{a3} & B^{a2} \\ E^{a2}/c & B^{a3} & 0 & -B^{a1} \\ E^{a3}/c & -B^{a2} & B^{a1} & 0 \end{bmatrix}. \tag{4.65}$$

We obtain the generalized Coulomb law by choosing $\nu = 0$:

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \mu_0 J^{a0}, \tag{4.66}$$

which, in vector notation, is

$$\nabla \cdot \mathbf{E}^a = c\,\mu_0 J^{a0}. \tag{4.67}$$

Because the 0-component of the current density is the charge density (see Eq. (4.53)) this equation can also be written as

$$\nabla \cdot \mathbf{E}^a = \frac{\rho^a}{\varepsilon_0}. \tag{4.68}$$

The Ampère-Maxwell law follows from choosing $\nu = 1,2,3$, giving three component equations:

$$\begin{aligned} \partial_0 F^{a01} + \partial_2 F^{a21} + \partial_3 F^{a31} &= \mu_0 J^{a1}, \\ \partial_0 F^{a02} + \partial_1 F^{a12} + \partial_3 F^{a32} &= \mu_0 J^{a2}, \\ \partial_0 F^{a03} + \partial_1 F^{a13} + \partial_2 F^{a23} &= \mu_0 J^{a3}, \end{aligned} \tag{4.69}$$

which with the aid of (4.65) become:

$$\begin{aligned} -\partial_0 E^{a01}/c + \partial_2 B^{a3} - \partial_3 B^{a2} &= \mu_0 J^{a1}, \\ -\partial_0 E^{a02}/c - \partial_1 B^{a3} + \partial_3 B^{a1} &= \mu_0 J^{a2}, \\ -\partial_0 E^{a03}/c + \partial_1 F^{a2} - \partial_2 B^{a1} &= \mu_0 J^{a3}. \end{aligned} \tag{4.70}$$

These can be condensed into one vector equation again:

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} + \nabla \times \mathbf{B}^a = \mu_0 \mathbf{J}^a. \tag{4.71}$$

Ultimately, we arrive at the Maxwell-like field equations, in vector form:

$$\nabla \cdot \mathbf{B}^a = -\mu_0 j^{a0}, \tag{4.72}$$

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = c\,\mu_0 \mathbf{j}^a, \tag{4.73}$$

$$\nabla \cdot \mathbf{E}^a = \frac{\rho^a}{\varepsilon_0}, \tag{4.74}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} + \nabla \times \mathbf{B}^a, = \mu_0 \mathbf{J}^a, \tag{4.75}$$

which correspond, according to Eqs. (4.13 - 4.16), to the Gauss law, the Faraday law, the Coulomb law and the Ampère-Maxwell law. These equations are valid in a generally covariant spacetime. Because of the four values for the polarization index a, the equation system consists of $4 \cdot 8 = 32$ equations. There are only $4 \cdot 6 = 24$ variables. It is known, however, that the Gauss law is dependent on the Faraday law, and the Coulomb law is dependent on the Ampère-Maxwell law. Therefore, there are only 24 independent equations, and the equation system is uniquely defined. In particular, the equations for each a index separate. The meaning of the polarization index will be clarified next through two detailed examples.

4.2.4 Examples of ECE field equations

The Coulomb law in Cartan geometry

■ **Example 4.1** In Chapters 2 and 3 we demonstrated, by examples, how all elements of a given tetrad can be calculated within Cartan geometry. Now we extend this method to physical fields.

One of the simplest and most important cases in electrodynamics is the Coulomb potential. In 4-vector notation, the potential is the 0-component

$$A^0 = \frac{\phi(r)}{c} = \frac{1}{c}\frac{q_e}{4\pi\varepsilon_0 r}, \tag{4.76}$$

where q_e is the central point charge and r is the radial coordinate of a spherical coordinate system

$$(X^\mu) = \begin{bmatrix} t \\ r \\ \theta \\ \phi \end{bmatrix}. \tag{4.77}$$

According to Eq. (4.2), the potential corresponds to the first diagonal element of the tetrad:

$$\phi(r) = c\, A^{(0)} q^{(0)}{}_0. \tag{4.78}$$

Inserting the potential into the q matrix gives

$$(q^a{}_\mu) = \frac{1}{2}\frac{(A^a{}_\mu)}{A^{(0)}} = \frac{1}{A^{(0)}} \begin{bmatrix} \frac{\phi(r)}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.79}$$

which is a singular matrix. Cartan Geometry, however, is only defined with non-singular tetrads (see Section 2.5.1). Therefore, a vector potential is necessarily required, in addition to a scalar potential. We choose the simplest form, a constant vector potential, which gives no magnetostatic field. The final form of the tetrad then is

$$(q^a{}_\mu) = \frac{1}{2} \begin{bmatrix} \frac{C_0}{r} & 0 & 0 & 0 \\ 0 & -C_1 & 0 & 0 \\ 0 & 0 & -C_2 & 0 \\ 0 & 0 & 0 & -C_3 \end{bmatrix}, \tag{4.80}$$

where

$$C_0 = \frac{q_e}{A^{(0)}\, c\, 4\pi\varepsilon_0} \tag{4.81}$$

and the C_i are arbitrary constants for $i = 1,2,3$. For simplicity of results, we assume $C_i > 0$ and omit the factors $A^{(0)}$ and c. Then, the vector potential is

$$\mathbf{A} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}. \tag{4.82}$$

Cartan geometry is now applied as follows (we repeat the relevant equations):

Metric compatibility (2.149):

$$D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda{}_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\sigma\nu} g_{\mu\lambda} = 0 \tag{4.83}$$

with an explicit antisymmetry requirement for all non-diagonal Γ elements (2.156):

$$\Gamma^\rho{}_{\mu\nu} = -\Gamma^\rho{}_{\nu\mu}. \tag{4.84}$$

The metric (2.204):

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu\, \eta_{ab}, \tag{4.85}$$

$$g^{\mu\nu} = \frac{1}{n}\, q^\mu{}_a q^\nu{}_b\, \eta^{ab}. \tag{4.86}$$

The spin connection (2.235):

$$\omega^{a}{}_{\mu b} = q^{a}{}_{\nu} q^{\lambda}{}_{b} \Gamma^{\nu}{}_{\mu\lambda} - q^{\lambda}{}_{b} \partial_{\mu} q^{a}{}_{\lambda}. \tag{4.87}$$

The Λ connection and its spin connection (3.48-3.49):

$$\Lambda^{\lambda}{}_{\mu\nu} = \frac{1}{2} |g|^{-1/2} \eta^{\rho\alpha} \eta^{\sigma\beta} \varepsilon_{\rho\sigma\mu\nu} \Gamma^{\lambda}{}_{\alpha\beta}, \tag{4.88}$$

$$\omega_{(\Lambda)}{}^{a}{}_{\mu b} = q^{a}{}_{\nu} q^{\lambda}{}_{b} \Lambda^{\nu}{}_{\mu\lambda} - q^{\lambda}{}_{b} \partial_{\mu} q^{a}{}_{\lambda}. \tag{4.89}$$

The torsion and curvature tensors:

$$R^{\lambda}{}_{\mu\nu\rho} = \partial_{\mu} \Gamma^{\lambda}_{\nu\rho} - \partial_{\nu} \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma} \Gamma^{\sigma}_{\mu\rho}, \tag{4.90}$$

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}, \tag{4.91}$$

and their forms:

$$R^{a}{}_{b\mu\nu} = q^{a}{}_{\rho} q^{\sigma}{}_{b} R^{\rho}{}_{\sigma\mu\nu}, \tag{4.92}$$

$$T^{a}{}_{\mu\nu} = q^{a}{}_{\lambda} T^{\lambda}{}_{\mu\nu}, \tag{4.93}$$

and contravariant forms:

$$R^{a}{}_{b}{}^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} R^{a}{}_{b\rho\sigma}, \tag{4.94}$$

$$T^{a\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} T^{a}{}_{\rho\sigma}. \tag{4.95}$$

Now we evaluate the equations with the tetrad (4.80) (the Maxima code is in [59]). This gives Γ connections with four unspecified parameters D_1 to D_4:

$$\begin{aligned}
\Gamma^{0}{}_{01} &= \frac{1}{r} \\
\Gamma^{0}{}_{10} &= -\frac{1}{r} \\
\Gamma^{0}{}_{12} &= \frac{D_4 {C_2}^2 r^2}{{C_0}^2} \\
\Gamma^{0}{}_{13} &= -\frac{D_3 {C_1}^2 r^2}{{C_0}^2} \\
&\dots
\end{aligned} \tag{4.96}$$

It is possible to set the D_i to zero:

$$D_1 = D_2 = D_3 = D_4 = 0. \tag{4.97}$$

Then, only three non-vanishing connections remain:

$$\Gamma^{0}{}_{01} = \frac{1}{r}, \tag{4.98}$$

$$\Gamma^{0}{}_{10} = -\frac{1}{r}, \tag{4.99}$$

$$\Gamma^{1}{}_{00} = \frac{C_0^2}{C_1^2 r^3}. \tag{4.100}$$

The first pair is antisymmetric, while the third connection is a diagonal element which does not contribute to torsion.

Applying Eq. (4.87), the non-vanishing spin connections are

$$\omega^{(0)}{}_{0(1)} = -\frac{C_0}{C_1 r^2}, \tag{4.101}$$

$$\omega^{(1)}{}_{0(0)} = -\frac{C_0}{C_1 r^2}, \tag{4.102}$$

which are antisymmetric in indices in a and b. (Please notice that the upper index a has to be lowered for comparison, which gives a sign change for the second connection element.) We have written the Latin indices in parentheses in order to distinguish these numbers from those stemming from Greek indices.

The Hodge duals of the Γ connection are

$$\Lambda^0{}_{23} = -\frac{1}{r}, \tag{4.103}$$

$$\Lambda^0{}_{32} = \frac{1}{r}, \tag{4.104}$$

and are complementary to the Γ's in the lower indices. The non-zero Λ spin connections are

$$\omega_{(\Lambda)}{}^{(0)}{}_{1(0)} = \frac{1}{r}, \tag{4.105}$$

$$\omega_{(\Lambda)}{}^{(0)}{}_{2(3)} = \frac{C_0}{C_3 r^2}, \tag{4.106}$$

$$\omega_{(\Lambda)}{}^{(0)}{}_{3(2)} = -\frac{C_0}{C_2 r^2}. \tag{4.107}$$

It is important to note that the connection $\omega_{(\Lambda)}{}^{(0)}{}_{1(0)}$ has the form that was derived in early papers of the UFT series. In those papers, the spin connections for Γ and Λ had not been discerned, and which one was meant depended on the field equations used. In the inhomogeneous current (Coulomb and Ampère-Maxwell laws), the Λ spin connections appear.

The non-vanishing torsion and curvature tensor elements are

$$T^0{}_{01} = -T^0{}_{10} = \frac{2}{r}, \tag{4.108}$$

$$R^0{}_{101} = -R^0{}_{110} = \frac{2}{r^2}, \tag{4.109}$$

$$R^1{}_{001} = -R^1{}_{010} = \frac{2C_0^2}{C_1^2 r^4}, \tag{4.110}$$

which are all antisymmetric in the last two indices. The same holds for the torsion and curvature forms:

$$T^{(0)}{}_{01} = -T^{(0)}{}_{10} = \frac{C_0}{r^2}, \tag{4.111}$$

$$R^{(0)}{}_{(1)01} = -R^{(0)}{}_{(1)10} = -\frac{2C_0}{C_1 r^3}, \tag{4.112}$$

$$R^{(1)}{}_{(0)01} = -R^{(1)}{}_{(0)10} = -\frac{2C_0}{C_1 r^3}. \tag{4.113}$$

The final results are obtained by inserting the torsion elements into Eq. (4.65) leading, e.g., to

$$F^{a01} = A^{(0)} T^{a01} = \frac{E^{a1}}{c}, \tag{4.114}$$

from which follows

$$E^{a1} = cA^{(0)}T^{a01}, \tag{4.115}$$

which is the r-component of the fields $\mathbf{E}^a$. The field elements for all other a values are obtained in the same way. This gives for the electric fields

$$\mathbf{E}^{(0)} = cA^{(0)} \begin{bmatrix} \frac{C_0}{r^2} \\ 0 \\ 0 \end{bmatrix}, \tag{4.116}$$

$$\mathbf{E}^{(1)} = \mathbf{E}^{(2)} = \mathbf{E}^{(3)} = \mathbf{0}, \tag{4.117}$$

and for the magnetic fields

$$\mathbf{B}^{(0)} = \mathbf{B}^{(1)} = \mathbf{B}^{(2)} = \mathbf{B}^{(3)} = \mathbf{0}. \tag{4.118}$$

Only the electric 0-component of polarization is not a zero vector, and all polarizations of the magnetic field vanish. This is exactly the classical result

$$\mathbf{E}^{(0)} = \mathbf{E} = \frac{q_e}{4\pi\varepsilon_0 r^2}. \tag{4.119}$$

It is seen that - despite the purely classical result - there are non-vanishing spin connections and curvature tensor elements. This shows that ECE theory gives results beyond classical electromagnetism. The latter is based on special relativity only. ∎

Circularly polarized plane wave in complex basis - the $\mathbf{B}^{(3)}$ field

∎ **Example 4.2** In the early 1990s, Myron Evans developed what is known as the B(3) field, a longitudinal field of electrodynamics [23, 24, 25] that describes a longitudinal component of electromagnetic waves. He then generalized this theory to O(3) electrodynamics [25, 26], in the late 1990s, and these two theories culminated in ECE theory, in 2003.

We will now present the basics of O(3) electrodynamics and show how circularly polarized plane waves (Figs. 4.1, 4.2) can be attributed to different polarization vectors of the electric and magnetic fields.

The usual orthonormal basis of the three-dimensional cartesian space is described by the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.120}$$

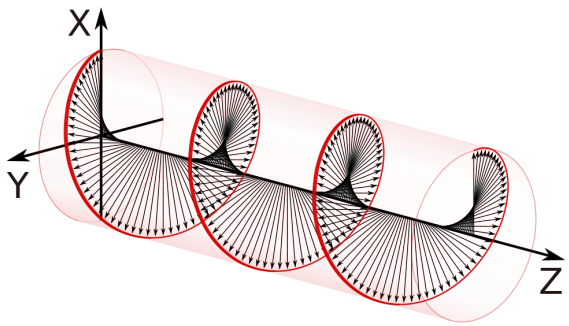

Figure 4.1: Right-polarized wave [68].

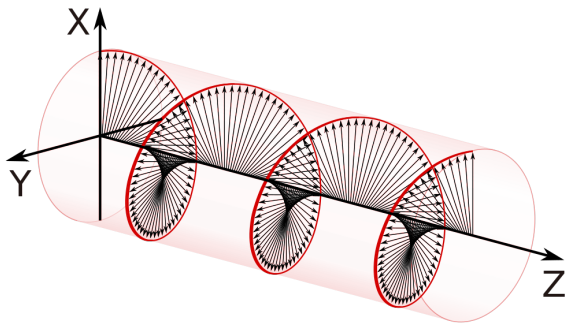

Figure 4.2: left-polarized wave [69].

These fulfill the circular relations

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}, \tag{4.121}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i}, \tag{4.122}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j}. \tag{4.123}$$

In O(3) electrodynamics, a complex circular basis in flat space is used, denoted by $\mathbf{q}^{(1)},\mathbf{q}^{(2)},\mathbf{q}^{(3)}$. The transformation equations from the cartesian to the complex circular basis are:

$$\mathbf{q}^{(1)}=\frac{1}{\sqrt{2}}(\mathbf{i}-i\mathbf{j}), \tag{4.124}$$
$$\mathbf{q}^{(2)}=\frac{1}{\sqrt{2}}(\mathbf{i}+i\mathbf{j}), \tag{4.125}$$
$$\mathbf{q}^{(3)}=\mathbf{k}, \tag{4.126}$$

where i is the imaginary unit (not to be confused with the unit vector $\mathbf{i}$). These vectors have a spacial circular relation, which is characteristic for O(3) symmetry:

$$\mathbf{q}^{(1)}\times\mathbf{q}^{(2)}=i\mathbf{q}^{(3)*}, \tag{4.127}$$
$$\mathbf{q}^{(2)}\times\mathbf{q}^{(1)}=i\mathbf{q}^{(1)*}, \tag{4.128}$$
$$\mathbf{q}^{(3)}\times\mathbf{q}^{(1)}=i\mathbf{q}^{(2)*}. \tag{4.129}$$

In contrast to (4.121 - 4.123), the cross products of two basis vectors do not lead directly to the third vector, but to the conjugate vector, multiplied by the imaginary unit. Eqs. (4.124 - 4.126) are basis transformations, therefore they can be interpreted directly as a Cartan tetrad $q^a{}_\mu$ where indices a and μ run from 1 to 3. Consequently, we can define vector potentials according to the first ECE axiom (4.11), multiplying the q's by the factor $A^{(0)}$:

$$\mathbf{A}^{(1)}=A^{(0)}\mathbf{q}^{(1)}, \tag{4.130}$$
$$\mathbf{A}^{(2)}=A^{(0)}\mathbf{q}^{(2)}, \tag{4.131}$$
$$\mathbf{A}^{(3)}=A^{(0)}\mathbf{q}^{(3)}. \tag{4.132}$$

In so doing, we have defined potentials for three polarization directions, where these directions coincide with the axes of the coordinate system. The polarization in Z direction is constant:

$$\mathbf{A}^{(3)}=A^{(0)}\mathbf{k}. \tag{4.133}$$

In the absence of rotation around Z, we have:

$$\nabla\times\mathbf{A}^{(1)}=\nabla\times\mathbf{A}^{(2)}=\mathbf{0}. \tag{4.134}$$

Now we define a wave of $\mathbf{A}$ vectors rotating in the XY plane:

$$\mathbf{A}^{(1)}=A^{(0)}\mathbf{q}^{(1)}e^{i(\omega t-\kappa Z)}, \tag{4.135}$$
$$\mathbf{A}^{(2)}=A^{(0)}\mathbf{q}^{(2)}e^{-i(\omega t-\kappa Z)}, \tag{4.136}$$
$$\mathbf{A}^{(3)}=A^{(0)}\mathbf{q}^{(3)}, \tag{4.137}$$

where ω is a time frequency, and κ is a wave number (the spatial frequency of a wave, measured in cycles per unit distance) in the Z direction. The first two $\mathbf{A}$ vectors define a left and right rotation around the Z axis. They are related by

$$\mathbf{A}^{(1)}=\mathbf{A}^{(2)*}. \tag{4.138}$$

$\mathbf{A}^{(3)}$ is a constant vector in the Z direction.

The magnetic field of O(3) electrodynamics is defined by

$$\mathbf{B}^{(1)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(2)} \times \mathbf{A}^{(3)}, \tag{4.139}$$

$$\mathbf{B}^{(2)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(3)} \times \mathbf{A}^{(1)}, \tag{4.140}$$

$$\mathbf{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}, \tag{4.141}$$

(notice that conjugated quantities are defined on the left-hand side). These fields obey the *B cyclic theorem*:

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iA^{(0)}\kappa\, \mathbf{B}^{(3)*}, \tag{4.142}$$

$$\mathbf{B}^{(2)} \times \mathbf{B}^{(3)} = iA^{(0)}\kappa\, \mathbf{B}^{(1)*}, \tag{4.143}$$

$$\mathbf{B}^{(3)} \times \mathbf{B}^{(1)} = iA^{(0)}\kappa\, \mathbf{B}^{(2)*}. \tag{4.144}$$

This is proven using computer algebra [60], along with many other theorems. It is also possible to define the first two polarizations in the same way as in conventional electrodynamics:

$$\mathbf{B}^{(1)} = \nabla \times \mathbf{A}^{(1)}, \tag{4.145}$$

$$\mathbf{B}^{(2)} = \nabla \times \mathbf{A}^{(2)}. \tag{4.146}$$

However, then there is no $\mathbf{B}^{(3)}$ field, because $\mathbf{A}^{(3)}$ is constant and therefore

$$\nabla \times \mathbf{A}^{(3)} = \mathbf{0}. \tag{4.147}$$

So $\mathbf{B}^{(3)}$ can only be defined by Eq. (4.141):

$$\mathbf{B}^{(3)} = i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)*} \times \mathbf{A}^{(2)*}. \tag{4.148}$$

The $\mathbf{B}^{(3)}$ field

$$\mathbf{B}^{(3)} = A^{(0)}\kappa\, \mathbf{k} \tag{4.149}$$

has been studied in great detail [23, 25]. It is a radiated magnetic field or flux density in the direction of wave propagation. Such a field is not known in ordinary electrodynamics. When the wave hits matter, it creates a magnetization in the propagation direction, which is known as the inverse Faraday effect. Besides this, there are other effects in spectroscopy that can be explained by the $\mathbf{B}^{(3)}$ field, namely in optical NMR and in laser technology [23].

Interestingly, there is no electrical $\mathbf{E}^{(3)}$ field. Electric field vectors of both left and right circular polarization can be defined by

$$\mathbf{E}^{(1)} = E^{(0)}\frac{1}{\sqrt{2}}(\mathbf{i} + i\mathbf{j})e^{i(\omega t - \kappa Z)}, \tag{4.150}$$

$$\mathbf{E}^{(2)} = E^{(0)}\frac{1}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})e^{-i(\omega t - \kappa Z)}, \tag{4.151}$$

and are perpendicular to the magnetic fields. The $\mathbf{E}^{(3)}$ field has to be the cross product of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, giving

$$\mathbf{E}^{(3)} = \frac{1}{E^{(0)}}\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = -iE^{(0)}\mathbf{k}, \tag{4.152}$$

with a suitable constant $E^{(0)}$, and is purely imaginary. Therefore, no physical $\mathbf{E}^{(3)}$ field exists (and has never been observed).

There is another interesting relation for circular plane waves. By comparison with the results of Eqs. (4.139 - 4.141) we find that

$$\mathbf{B}^{(1)} = \kappa_1 \mathbf{A}^{(1)}, \tag{4.153}$$

$$\mathbf{B}^{(2)} = \kappa_2 \mathbf{A}^{(2)}, \tag{4.154}$$

$$\mathbf{B}^{(3)} = \kappa_3 \mathbf{A}^{(3)}, \tag{4.155}$$

which, by comparison with (4.145 - 4.147), gives three *Beltrami conditions*:

$$\nabla \times \mathbf{A}^{(1)} = \kappa \mathbf{A}^{(1)}, \tag{4.156}$$

$$\nabla \times \mathbf{A}^{(2)} = \kappa \mathbf{A}^{(2)}, \tag{4.157}$$

$$\nabla \times \mathbf{A}^{(3)} = 0 \cdot \mathbf{A}^{(3)}. \tag{4.158}$$

These have to do with longitudinal waves and will be discussed later in this book. All calculations in this example can be verified by computer algebra code [60]. ∎

4.3 The wave equation

It was shown in Section 2.5.5 that a wave equation can be derived from the tetrad postulate. This is called the Evans lemma, Eq. (2.258):

$$\Box q^a{}_\nu + R q^a{}_\nu = 0. \tag{4.159}$$

This is an equation for the tetrad and contains a scalar curvature R, which, according to Eq. (2.257), is defined by the tetrad, spin connection and Γ connection terms of Cartan geometry:

$$R = q^\nu{}_a \left(\partial^\mu (\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu (\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda) \right). \tag{4.160}$$

The Evans lemma can easily be transformed into a physical equation by applying the first ECE postulate (4.11), i.e., multiplying the equation by the constant $A^{(0)}$, to obtain physical units of a potential:

$$\Box A^a{}_\nu + R A^a{}_\nu = 0. \tag{4.161}$$

The d'Alembert operator $\Box$ was already introduced in Section 2.5.5. In cartesian coordinates it reads:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2}. \tag{4.162}$$

Eq. (4.161) is different from the wave equation of standard electrodynamics, which is

$$\Box A^a{}_\nu = 0. \tag{4.163}$$

This equation has no curvature term and follows from (4.161) by $R = 0$, which indicates in a flat space without curvature and torsion. When the potential $A^a{}_\nu$ is derived as a solution of this equation, it is not uniquely determined. $A^a{}_\nu$ may be changed by a tensorial function $\phi^a{}_\nu$, whose derivatives vanish:

$$\Box \phi^a{}_\nu = 0, \tag{4.164}$$

or, in a particular case,

$$\frac{\partial \phi^a{}_\nu}{\partial t} = 0 \quad \text{and} \qquad (4.165)$$
$$\nabla^2\, \phi^a{}_\nu = 0,$$

so that

$$\Box\, (A^a{}_\nu + \phi^a{}_\nu) = 0. \qquad (4.166)$$

The particular case is called a *gauge operation*, which are the basis for quantum electrodynamics. When we use the ECE wave equation (4.161) instead, a re-gauging of the vector potential is no longer possible. Therefore, quantum electrodynamics is obsolete in the framework of ECE theory. All well-known effects of quantum electrodynamics, for example the Lamb shift in atomic spectra, can be explained by ECE theory directly, as a consequence of the fact that spacetime is not flat.

As a preview of ECE quantum mechanics, we state that the wave equation (4.161) can be quantized and is the basis of the Fermion equation, which is comparable to the Dirac equation in establishing relativistic quantum mechanics. The Fermion equation is not based on special relativity, but general relativity, a spacetime with curvature and torsion. The non-relativistic Schrödinger equation can be derived as an approximation of the Fermion equation, or, alternatively, from Einstein's relation for total energy. Thus, the ECE wave equation provides the foundation for all of ECE quantum mechanics.

The curvature term R in the ECE wave equation describes a coupling between spacetime or even gravitation and electromagnetism. Therefore, R may be replaced by a mass term and a dimensional factor. In order to obtain curvature units $(1/\mathrm{m}^2)$, we replace R by $(m_0 c/\hbar)^2$, where m_0 may be constant. This ensures that the dimensions are right, and we can write

$$\Box A^a{}_\nu + \left(\frac{m_0 c}{\hbar}\right)^2 A^a{}_\nu = 0. \qquad (4.167)$$

When applied to electromagnetic waves, it follows that photons have a rest mass m_0. This equation is identical to the Proca equation [27], which was derived by Proca independently before the advent of ECE theory. In the standard model, the Proca equation is directly incompatible with gauge invariance. The gauge principle is not tenable in a unified field theory such as ECE because the potential in ECE is physically relevant and cannot be "re-gauged". In ECE theory the tetrad postulate is invariant under the general coordinate transform, and this is the principle that governs the potential field in ECE.

4.4 Field equations in terms of potentials

The field equations of ECE theory are formally identical to Maxwell's equations, augmented by a polarization index, and are valid in a spacetime of general relativity. The connections of spacetime are not directly visible in this representation for the electric and magnetic force fields. However, we can make the underlying structure evident by replacing the force fields with their potentials. Because the force fields are identical with Cartan torsion, within a factor, we use the first Maurer-Cartan structure relation to express the force fields by their potentials and connections. The first Maurer-Cartan structure, Eq. (2.283), is the 2-form:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b, \qquad (4.168)$$

or written in tensor form:

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu. \qquad (4.169)$$

Applying the ECE axioms

$$A^a{}_\mu = A^{(0)} q^a{}_\mu, \tag{4.170}$$

$$F^a{}_{\mu\nu} = A^{(0)} T^a{}_{\mu\nu}, \tag{4.171}$$

then leads to the equation

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + \omega^a{}_{\mu b} A^b{}_\nu - \omega^a{}_{\nu b} A^b{}_\mu. \tag{4.172}$$

It is directly seen that this is an antisymmetric tensor consisting of the potential A and spin connection terms. In classical electrodynamics, the field tensor $F_{\mu\nu}$ is defined from the potentials as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.173}$$

There is no polarization index and there are no spin connection terms, indicating that this is an equation of Minkowski space without curved spacetime. Eq. (4.172) is a generalized version of this equation, introducing curvature and torsion.

In the preceding section we have introduced the vector representation of the ECE field equations. Therefore, it is useful to replace the vectors $\mathbf{E}^a$ and $\mathbf{B}^a$ by their potentials according to Eq. (4.172). In classical physics, we have

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.174}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4.175}$$

where ϕ is the electric scalar potential and $\mathbf{A}$ is the magnetic vector potential. The mapping between the elements of F and the electric and magnetic field was given by Eq. (4.65), but Eq. (4.56) can also be used, if the Hodge dual of F is considered. Here we use Eq. (4.65):

$$\mathbf{E}^a = \begin{bmatrix} E^{a1} \\ E^{a2} \\ E^{a3} \end{bmatrix} = -c \begin{bmatrix} F^{a01} \\ F^{a02} \\ F^{a03} \end{bmatrix}, \tag{4.176}$$

$$\mathbf{B}^a = \begin{bmatrix} B^{a1} \\ B^{a2} \\ B^{a3} \end{bmatrix} = \begin{bmatrix} F^{a32} \\ F^{a13} \\ F^{a21} \end{bmatrix}. \tag{4.177}$$

Then this follows from Eq. (4.172) for $\mu = 0,\ \nu = 1,2,3$:

$$F^a{}_{01} = \partial_0 A^a{}_1 - \partial_1 A^a{}_0 + \omega^a{}_{0b} A^b{}_1 - \omega^a{}_{1b} A^b{}_0, \tag{4.178}$$

$$F^a{}_{02} = \partial_0 A^a{}_2 - \partial_2 A^a{}_0 + \omega^a{}_{0b} A^b{}_2 - \omega^a{}_{2b} A^b{}_0, \tag{4.179}$$

$$F^a{}_{03} = \partial_0 A^a{}_3 - \partial_3 A^a{}_0 + \omega^a{}_{0b} A^b{}_3 - \omega^a{}_{3b} A^b{}_0. \tag{4.180}$$

We raise Greek indices in F, A and ω, which gives sign changes for $\nu = 1,2,3$:

$$-F^{a01} = -\partial_0 A^{a1} - \partial_1 A^{a0} - \omega^a{}_{0b} A^{b1} + \omega^{a1}{}_b A^{b0}, \tag{4.181}$$

$$-F^{a02} = -\partial_0 A^{a2} - \partial_2 A^{a0} - \omega^a{}_{0b} A^{b2} + \omega^{a2}{}_b A^{b0}, \tag{4.182}$$

$$-F^{a03} = -\partial_0 A^{a3} - \partial_3 A^{a0} - \omega^a{}_{0b} A^{b3} + \omega^{a3}{}_b A^{b0}. \tag{4.183}$$

Inserting (4.176), we obtain:

$$\frac{1}{c}E^{a1} = -\partial_0 A^{a1} - \partial_1 A^{a0} - \omega^a{}_{0b} A^{b1} + \omega^{a1}{}_b A^{b0}, \tag{4.184}$$

$$\frac{1}{c}E^{a2} = -\partial_0 A^{a2} - \partial_2 A^{a0} - \omega^a{}_{0b} A^{b2} + \omega^{a2}{}_b A^{b0}, \tag{4.185}$$

$$\frac{1}{c}E^{a3} = -\partial_0 A^{a3} - \partial_3 A^{a0} - \omega^a{}_{0b} A^{b3} + \omega^{a3}{}_b A^{b0}. \tag{4.186}$$

With

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \quad \text{and} \quad A^{a0} = \frac{\phi^a}{c} \tag{4.187}$$

this follows in vector form:

$$\mathbf{E}^a = -\nabla\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{A}^b + \omega^a{}_b\phi^b. \tag{4.188}$$

Please notice that $\omega^{a0}{}_b$ is a scalar and $\omega^a{}_b$ is a vector:

$$\omega^a{}_b = \begin{bmatrix} \omega^{a1}{}_b \\ \omega^{a2}{}_b \\ \omega^{a3}{}_b \end{bmatrix}. \tag{4.189}$$

The representation of the magnetic field vector is found from Eq. (4.177), using the corresponding combinations of μ and ν:

$$F^a{}_{32} = \partial_3 A^a{}_2 - \partial_2 A^a{}_3 + \omega^a{}_{3b} A^b{}_2 - \omega^a{}_{2b} A^b{}_3, \tag{4.190}$$

$$F^a{}_{13} = \partial_1 A^a{}_3 - \partial_3 A^a{}_1 + \omega^a{}_{1b} A^b{}_3 - \omega^a{}_{3b} A^b{}_1, \tag{4.191}$$

$$F^a{}_{21} = \partial_2 A^a{}_1 - \partial_1 A^a{}_2 + \omega^a{}_{2b} A^b{}_1 - \omega^a{}_{1b} A^b{}_2. \tag{4.192}$$

With Greek indices raised, we obtain

$$F^{a32} = -\partial_3 A^{a2} + \partial_2 A^{a3} + \omega^{a3}{}_b A^{b2} - \omega^{a2}{}_b A^{b3}, \tag{4.193}$$

$$F^{a13} = -\partial_1 A^{a3} + \partial_3 A^{a1} + \omega^{a1}{}_b A^{b3} - \omega^{a3}{}_b A^{b1}, \tag{4.194}$$

$$F^{a21} = -\partial_2 A^{a1} + \partial_1 A^{a2} + \omega^{a2}{}_b A^{b1} - \omega^{a1}{}_b A^{b2}. \tag{4.195}$$

In vector form this can be written as

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a - \omega^a{}_b \times \mathbf{A}^b. \tag{4.196}$$

The first Maurer-Cartan structure equation leads to the complete set of field-potential relations in ECE electrodynamics:

$$\boxed{\begin{aligned} \mathbf{E}^a &= -\nabla\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{A}^b + \omega^a{}_b\phi^b, \qquad (4.197)\\ \mathbf{B}^a &= \nabla \times \mathbf{A}^a - \omega^a{}_b \times \mathbf{A}^b. \qquad (4.198) \end{aligned}}$$

Inserting these relations into the field equations (4.72 - 4.75) leads to

$$\nabla\cdot\left(\nabla\times\mathbf{A}^a-\omega^a{}_b\times\mathbf{A}^b\right)=-\mu_0 j^{a0} \tag{4.199}$$

$$\frac{\partial\left(\nabla\times\mathbf{A}^a-\omega^a{}_b\times\mathbf{A}^b\right)}{\partial t}$$
$$+\nabla\times\left(-\nabla\phi^a-\frac{\partial\mathbf{A}^a}{\partial t}-c\omega^a{}_{0b}\mathbf{A}^b+\omega^a{}_b\phi^b\right)=c\,\mu_0\mathbf{j}^a, \tag{4.200}$$

$$\nabla\cdot\left(-\nabla\phi^a-\frac{\partial\mathbf{A}^a}{\partial t}-c\omega^a{}_{0b}\mathbf{A}^b+\omega^a{}_b\phi^b\right)=\frac{\rho^a}{\varepsilon_0}, \tag{4.201}$$

$$-\frac{1}{c^2}\frac{\partial\left(-\nabla\phi^a-\frac{\partial\mathbf{A}^a}{\partial t}-c\omega^a{}_{0b}\mathbf{A}^b+\omega^a{}_b\phi^b\right)}{\partial t}$$
$$+\nabla\times\left(\nabla\times\mathbf{A}^a-\omega^a{}_b\times\mathbf{A}^b\right)=\mu_0\mathbf{J}^a. \tag{4.202}$$

These equations can be simplified by using the theorems of vector algebra, giving:

$$\nabla\cdot\left(\omega^a{}_b\times\mathbf{A}^b\right)=\mu_0 j^{a0} \tag{4.203}$$

$$-c\nabla\times\left(\omega^a{}_{0b}\mathbf{A}^b\right)+\nabla\times\left(\omega^a{}_b\phi^b\right)-\frac{\partial\left(\omega^a{}_b\times\mathbf{A}^b\right)}{\partial t}=c\,\mu_0\mathbf{j}^a, \tag{4.204}$$

$$\nabla\cdot\frac{\partial\mathbf{A}^a}{\partial t}+\nabla^2\phi^a+c\nabla\cdot\left(\omega^a{}_{0b}\mathbf{A}^b\right)-\nabla\cdot\left(\omega^a{}_b\phi^b\right)=-\frac{\rho^a}{\varepsilon_0}, \tag{4.205}$$

$$\nabla\left(\nabla\cdot\mathbf{A}^a\right)-\nabla^2\mathbf{A}^a-\nabla\times\left(\omega^a{}_b\times\mathbf{A}^b\right)$$
$$+\frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}^a}{\partial t^2}+c\frac{\partial\left(\omega^a{}_{0b}\mathbf{A}^b\right)}{\partial t}+\nabla\frac{\partial\phi^a}{\partial t}-\frac{\partial\left(\omega^a{}_b\phi^b\right)}{\partial t}\right)=\mu_0\mathbf{J}^a. \tag{4.206}$$

These are the field equations in potential form. They are much more complicated than those written in terms of force fields. In the standard case of vanishing magnetic monopoles, we have $j^{a0}=0$, $\mathbf{j}^a=\mathbf{0}$, as usual. If there are no spin connections, the first two laws result in zero terms at the left-hand side, indicating that non-vanishing magnetic monopoles are only possible in a spacetime of general relativity.

If no polarization is present, we can omit the corresponding Latin indices. In this case, we only have one scalar and one vector potential, and one scalar and one vector spin connection:

$$\phi^a\to\phi, \tag{4.207}$$
$$\mathbf{A}^a\to\mathbf{A}, \tag{4.208}$$
$$\omega^a{}_{0b}\to\omega_0, \tag{4.209}$$
$$\omega^a{}_b\to\omega. \tag{4.210}$$

Then, Eqs. (4.203 - 4.206) simplify to

$$\nabla\cdot(\omega\times\mathbf{A})=\mu_0 j^0, \tag{4.211}$$

$$-c\nabla\times(\omega_0\mathbf{A})+\nabla\times(\omega\phi)-\frac{\partial(\omega\times\mathbf{A})}{\partial t}=c\,\mu_0\mathbf{j}, \tag{4.212}$$

$$\nabla\cdot\frac{\partial\mathbf{A}}{\partial t}+\Delta\phi+c\nabla\cdot(\omega_0\mathbf{A})-\nabla\cdot(\omega\phi)=-\frac{\rho}{\varepsilon_0}, \tag{4.213}$$

$$\nabla(\nabla\cdot\mathbf{A})-\Delta\mathbf{A}-\nabla\times(\omega\times\mathbf{A})$$
$$+\frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}}{\partial t^2}+c\frac{\partial(\omega_0\mathbf{A})}{\partial t}+\nabla\frac{\partial\phi}{\partial t}-\frac{\partial(\omega\phi)}{\partial t}\right)=\mu_0\mathbf{J}. \tag{4.214}$$

These are 8 component equations for 8 potential and spin connection variables. Formally, this equation system is uniquely defined, but the Gauss law is not independent from the Faraday law, and the Coulomb law is not independent from the Ampère-Maxwell law. This has to be taken into account when solving the equation system. The solutions become unique (the equations become independent) when the charge density and the current density are chosen in an unrelated way [31].

5. Advanced properties of electrodynamics

After having introduced ECE electrodynamics in Chapter 4, we complete the topic here by discussing special features and showing, through detailed examples, that they can be derived directly and easily with ECE theory, but not at all (or only with difficulty and inconsistencies) using standard physics.

5.1 The Antisymmetry laws

As shown in Section 2.5.4, the tetrad postulate can be written in form of Eq. (2.236) (with index renaming):

$$q^a{}_\nu \Gamma^\nu{}_{\mu\nu} = \partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b}. \tag{5.1}$$

We can define a mixed-index Γ connection by

$$\Gamma^a{}_{\mu\lambda} := q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b} \tag{5.2}$$

so that the antisymmetric Cartan torsion can be written as

$$T^a{}_{\mu\nu} = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu}. \tag{5.3}$$

In Chapter 2 we have found that the Γ connection may contain non-vanishing elements for $\mu = \nu$ but only the antisymmetric parts for $\mu \neq \nu$ are relevant. For a priori antisymmetric non-diagonal elements of Γ we have

$$\Gamma^a{}_{\mu\nu} = -\Gamma^a{}_{\nu\mu} \tag{5.4}$$

so that for the torsion form (5.3) it follows that:

$$T^a{}_{\mu\nu} = 2\,\Gamma^a{}_{\mu\nu} = 2\left(\partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b}\right). \tag{5.5}$$

On the other hand, torsion can be written according to Eq. (4.169) as

$$T^{a}_{\mu\nu} = \partial_\mu q^{a}{}_{\nu} - \partial_\nu q^{a}{}_{\mu} + \omega^{a}{}_{\mu b} q^{b}{}_{\nu} - \omega^{a}{}_{\nu b} q^{b}{}_{\mu}. \tag{5.6}$$

Equating both expressions for $T^{a}_{\mu\nu}$ then gives the relation

$$2\left(\partial_\mu q^{a}{}_{\nu} + q^{b}{}_{\nu}\omega^{a}{}_{\mu b}\right) = \partial_\mu q^{a}{}_{\nu} - \partial_\nu q^{a}{}_{\mu} + \omega^{a}{}_{\mu b} q^{b}{}_{\nu} - \omega^{a}{}_{\nu b} q^{b}{}_{\mu}, \tag{5.7}$$

which can be rearranged to

$$\boxed{\partial_\mu q^{a}{}_{\nu} + \partial_\nu q^{a}{}_{\mu} + \omega^{a}{}_{\mu b} q^{b}{}_{\nu} + \omega^{a}{}_{\nu b} q^{b}{}_{\mu} = 0.} \tag{5.8}$$

This is the *antisymmetry condition* of ECE theory.

Next, we will discuss how this impacts the vector notation of **E** and **B** fields. Applying the ECE axioms (4.170 - 4.171) gives

$$\partial_\mu A^{a}{}_{\nu} + \partial_\nu A^{a}{}_{\mu} + \omega^{a}{}_{\mu b} A^{b}{}_{\nu} + \omega^{a}{}_{\nu b} A^{b}{}_{\mu} = 0. \tag{5.9}$$

For $\mu = 0, \nu = 1,2,3$ we obtain

$$\partial_0 A^{a}{}_{1} + \partial_1 A^{a}{}_{0} + \omega^{a}{}_{0b} A^{b}{}_{1} + \omega^{a}{}_{1b} A^{b}{}_{0} = 0, \tag{5.10}$$
$$\partial_0 A^{a}{}_{2} + \partial_2 A^{a}{}_{0} + \omega^{a}{}_{0b} A^{b}{}_{2} + \omega^{a}{}_{2b} A^{b}{}_{0} = 0, \tag{5.11}$$
$$\partial_0 A^{a}{}_{3} + \partial_3 A^{a}{}_{0} + \omega^{a}{}_{0b} A^{b}{}_{3} + \omega^{a}{}_{3b} A^{b}{}_{0} = 0. \tag{5.12}$$

The Greek indices of A and ω can be raised with sign change for $\nu = 1,2,3$. In vector notation this gives

$$-\frac{1}{c}\frac{\partial \mathbf{A}^a}{\partial t} + \nabla A^{a}{}_{0} - \omega^{a}{}_{0b}\mathbf{A}^b - \omega^{a}{}_{b} A^{b}{}_{0} = \mathbf{0}, \tag{5.13}$$

or, with $A^{a}{}_{0} = \phi^a/c$,

$$\boxed{-\frac{\partial \mathbf{A}^a}{\partial t} + \nabla\phi^a - c\omega^{a}{}_{0b}\mathbf{A}^b - \omega^{a}{}_{b}\phi^b = \mathbf{0}.} \tag{5.14}$$

These are the *electric antisymmetry conditions*, because the terms of the ECE electric field appear. For $\mu \neq 0$ we obtain from (5.9):

$$\partial_3 A^{a}{}_{2} + \partial_2 A^{a}{}_{3} + \omega^{a}{}_{3b} A^{b}{}_{2} + \omega^{a}{}_{2b} A^{b}{}_{3} = 0, \tag{5.15}$$
$$\partial_1 A^{a}{}_{3} + \partial_3 A^{a}{}_{1} + \omega^{a}{}_{1b} A^{b}{}_{3} + \omega^{a}{}_{3b} A^{b}{}_{1} = 0, \tag{5.16}$$
$$\partial_2 A^{a}{}_{1} + \partial_1 A^{a}{}_{2} + \omega^{a}{}_{2b} A^{b}{}_{1} + \omega^{a}{}_{1b} A^{b}{}_{2} = 0, \tag{5.17}$$

and with indices raised:

$$\boxed{\begin{aligned} -\partial_3 A^{a2} - \partial_2 A^{a3} + \omega^{a3}{}_{b} A^{b2} + \omega^{a2}{}_{b} A^{b3} = 0, \quad (5.18)\\ -\partial_1 A^{a3} - \partial_3 A^{a1} + \omega^{a1}{}_{b} A^{b3} + \omega^{a3}{}_{b} A^{b1} = 0, \quad (5.19)\\ -\partial_2 A^{a1} - \partial_1 A^{a2} + \omega^{a2}{}_{b} A^{b1} + \omega^{a1}{}_{b} A^{b2} = 0. \quad (5.20) \end{aligned}}$$

These equations are called the *magnetic antisymmetry conditions*, because they relate to the magnetic vector potential $\mathbf{A}^a$. These equations have a permutational structure and cannot be written in vector form.

The antisymmetry conditions are constraints for the fields $\mathbf{E}^a$ and $\mathbf{B}^a$. Therefore, Equations (4.188) and (4.196) can be reformulated. First, let us use Eq. (5.5) directly. With the ECE axioms, we can write

$$\begin{aligned} F^a{}_{\mu\nu} &= 2A^{(0)}\,\Gamma^a{}_{\mu\nu} = 2A^{(0)}\left(\partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b}\right) \\ &= 2\left(\partial_\mu A^a{}_\nu + \omega^a{}_{\mu b} A^b{}_\nu\right). \end{aligned} \tag{5.21}$$

For $\mu = 0$ we obtain the electric field:

$$\begin{aligned} \mathbf{E}^a &= -\begin{bmatrix} F^{a01} \\ F^{a02} \\ F^{a03} \end{bmatrix} = \begin{bmatrix} F^a{}_{01} \\ F^a{}_{02} \\ F^a{}_{03} \end{bmatrix} = 2c \begin{bmatrix} \partial_0 A^a{}_1 + \omega^a{}_{0b} A^b{}_1 \\ \partial_0 A^a{}_2 + \omega^a{}_{0b} A^b{}_2 \\ \partial_0 A^a{}_3 + \omega^a{}_{0b} A^b{}_3 \end{bmatrix} = -2c \begin{bmatrix} \partial_0 A^{a1} + \omega^a{}_{0b} A^{b1} \\ \partial_0 A^{a2} + \omega^a{}_{0b} A^{b2} \\ \partial_0 A^{a3} + \omega^a{}_{0b} A^{b3} \end{bmatrix} \\ &= -2\left(\frac{\partial \mathbf{A}^a}{\partial t} + c\omega^a{}_{0b}\mathbf{A}^b.\right) \end{aligned} \tag{5.22}$$

For the magnetic field we obtain:

$$\begin{aligned} \mathbf{B}^a &= \begin{bmatrix} F^{a32} \\ F^{a13} \\ F^{a21} \end{bmatrix} = \begin{bmatrix} F^a{}_{32} \\ F^a{}_{13} \\ F^a{}_{21} \end{bmatrix} = 2 \begin{bmatrix} \partial_3 A^a{}_2 + \omega^a{}_{2b} A^b{}_3 \\ \partial_1 A^a{}_3 + \omega^a{}_{3b} A^b{}_1 \\ \partial_2 A^a{}_1 + \omega^a{}_{1b} A^b{}_2 \end{bmatrix} \\ &= 2 \begin{bmatrix} -\partial_3 A^{a2} + \omega^{a2}{}_b A^{b3} \\ -\partial_1 A^{a3} + \omega^{a3}{}_b A^{b1} \\ -\partial_2 A^{a1} + \omega^{a1}{}_b A^{b2} \end{bmatrix} \end{aligned} \tag{5.23}$$

This equation cannot be written in form of vector operators.

The electric antisymmetry condition (5.14) can be used to replace the two terms containing the $\mathbf{A}^a$ field with terms of the potential ϕ^a. Inserting this into Eq. (5.22), we obtain two formulations for the electric field vector:

$$\mathbf{E}^a = -2\left(\frac{\partial \mathbf{A}^a}{\partial t} + c\omega^a{}_{0b}\mathbf{A}^b\right) = -2\left(\nabla\phi^a - \omega^a{}_b\phi^b\right). \tag{5.24}$$

This is a remarkable result. The electric field is either defined by either the vector potential or the scalar potential, in combination with the scalar and vector spin connections. There is no counterpart in classical electrodynamics. If we omitted the spin connections, we would have $\frac{\partial \mathbf{A}^a}{\partial t} = \nabla\phi^a$, which is not generally true in Maxwellian electrodynamics.

Finally, we can also rewrite the ECE magnetic field (5.23) by means of the magnetic antisymmetry conditions (5.18 - 5.20):

$$\mathbf{B}^a = 2 \begin{bmatrix} -\partial_3 A^{a2} + \omega^{a2}{}_b A^{b3} \\ -\partial_1 A^{a3} + \omega^{a3}{}_b A^{b1} \\ -\partial_2 A^{a1} + \omega^{a1}{}_b A^{b2} \end{bmatrix} = 2 \begin{bmatrix} \partial_2 A^{a3} - \omega^{a3}{}_b A^{b2} \\ \partial_3 A^{a1} - \omega^{a1}{}_b A^{b3} \\ \partial_1 A^{a2} - \omega^{a2}{}_b A^{b1} \end{bmatrix}. \tag{5.25}$$

This is simply the application of an antisymmetry operation. The factor of 2 appears in Eqs. (5.24) and (5.25) for the "missing terms" when compared with the original definitions (4.197, 4.198).

■ **Example 5.1** In this example we show that classical electrodynamics, which uses U(1) symmetry, is not compatible with the antisymmetry laws of ECE theory. The antisymmetric field tensor is defined in U(1) symmetry [28]-[31] by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5.26}$$

and the antisymmetry of this definition requires

$$\partial_\mu A_\nu = -\partial_\nu A_\mu. \tag{5.27}$$

There is no polarization index and there are no spin connection terms. The electric and magnetic field vectors, in terms of potentials, have the well-known form:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{5.28}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{5.29}$$

From the antisymmetry law (5.27) it follows that

$$\nabla\phi = \frac{\partial \mathbf{A}}{\partial t}, \tag{5.30}$$

then, because the curl of a gradient field vanishes:

$$\nabla \times \nabla\phi = \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \mathbf{0}. \tag{5.31}$$

Therefore:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{5.32}$$

From Eq. (5.31) it follows that

$$\nabla \times \mathbf{E} = \mathbf{0}. \tag{5.33}$$

On the other hand, the Faraday law is

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{5.34}$$

If $\mathbf{E}$ were a time-dependent field, we would have $\frac{\partial \mathbf{B}}{\partial t} \neq \mathbf{0}$; therefore, $\mathbf{E}$ must be a static field. From the antisymmetry equation (5.30) it follows that

$$\nabla\phi = \frac{\partial \mathbf{A}}{\partial t} = \mathbf{0} \tag{5.35}$$

and so, in particular for a static electric field,

$$\mathbf{E} = -\nabla\phi = \mathbf{0}. \tag{5.36}$$

In standard theory it is assumed $\mathbf{A} = \mathbf{0}$ for static electric fields. Therefore, we have to exacerbate Eq. (5.32) to

$$\mathbf{B} = \mathbf{0}. \tag{5.37}$$

This has severe consequences that are described in [29] as follows:

The catastrophic result is obtained that the [static] E and B fields vanish on the U(1) level. All attempts at constructing a unified field theory based on a U(1) sector symmetry are incorrect fundamentally. Even worse for the standard physics is that the method introduced by Heaviside of expressing electric and magnetic fields through Eqs. (5.28) and (5.29) must be abandoned, so all of twentieth century gauge theory is proven to be empty dogma. This conclusion reinforces many other ways of showing that a U(1) gauge theory of electromagnetism is incorrect and that gauge freedom in the natural sciences is an illusion.

Here no gauge freedom means that the potential cannot be shifted arbitrarily, because it has a physical meaning. ∎

5.2 Polarization and Magnetization

5.2.1 Derivation from standard theory

Standard electrodynamics theory has been extended to media which are polarizable by electric fields and magnetizable by magnetic fields. These material properties evoke additional fields in the media, polarization **P** and magnetization **M**. The resulting total electric field is the dielectric displacement

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{5.38}$$

For magnetic materials, the induction is the sum of the magnetic field **H** and magnetization **M**:

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \tag{5.39}$$

ε_0 is the vacuum permittivity and μ_0 the vacuum permeability. They are related by the velocity of light in vacuo c:

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}. \tag{5.40}$$

In the case of isotropic materials with linear polarization/magnetization properties, the material fields depend linearly on the electric and magnetic fields and can be written

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}, \tag{5.41}$$
$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}, \tag{5.42}$$

where ε_r is the relative permittivity and μ_r is the relative permeability. In vacuo:

$$\varepsilon_r = 1, \quad \mu_r = 1. \tag{5.43}$$

The material equations (5.41, 5.42) can be generalized to ECE equations in a spacetime of general relativity, as we have done for the **E** and **B** fields. Since these are linear relations, the displacement and magnetic field can be augmented by an ECE polarization index a[1]:

$$\mathbf{D}^a = \varepsilon_0 \varepsilon_r \mathbf{E}^a, \tag{5.44}$$
$$\mathbf{B}^a = \mu_0 \mu_r \mathbf{H}^a. \tag{5.45}$$

The Faraday law in vacuo

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = \mathbf{0} \tag{5.46}$$

can be rewritten with aid of (5.44, 5.45) and $\varepsilon_r = 1, \mu_r = 1$ to

$$\frac{1}{c^2} \frac{\partial \mathbf{H}^a}{\partial t} + \nabla \times \mathbf{D}^a = \mathbf{0}. \tag{5.47}$$

In matter, the **H** and **D** fields are changed according to Eqs. (5.38, 5.39). Using the simplified relations (5.41, 5.42), we can express these fields by the vacuum fields **E** and **B** (but we would have to use different variable names to be fully correct). Introducing the a index as before, we obtain:

$$\frac{1}{\mu_r} \frac{\partial \mathbf{B}^a}{\partial t} + \varepsilon_r \nabla \times \mathbf{E}^a = \mathbf{0} \tag{5.48}$$

[1] Please notice that this "polarization" is a spacetime property and does not have anything to do with dielectric polarization.

which is an alternative version of the Faraday law in matter. For the Ampère-Maxwell law, we obtain (in the same way):

$$-c^2\frac{\partial \mathbf{D}^a}{\partial t}+\nabla\times\mathbf{H}^a=\mathbf{J}^a \tag{5.49}$$

and

$$-\varepsilon_r\frac{\partial \mathbf{E}^a}{\partial t}+\frac{1}{\mu_r}\nabla\times\mathbf{B}^a=\mu_0\mathbf{J}^a, \tag{5.50}$$

where $\mathbf{J}^a$ is a "free" external current, independent of polarization and magnetization. The Gauss law remains as is and the Coulomb law becomes

$$\nabla\cdot\mathbf{E}^a=\frac{\rho^a}{\varepsilon_0\varepsilon_r}. \tag{5.51}$$

Ultimately, we arrive at the ECE field equations for polarizable and magnetizable materials, in vector form:

$$\nabla\cdot\mathbf{B}^a=0, \tag{5.52}$$

$$\frac{1}{\mu_r}\frac{\partial \mathbf{B}^a}{\partial t}+\varepsilon_r\nabla\times\mathbf{E}^a=\mathbf{0}, \tag{5.53}$$

$$\nabla\cdot\mathbf{E}^a=\frac{\rho^a}{\varepsilon_0\varepsilon_r}, \tag{5.54}$$

$$-\varepsilon_r\frac{\partial \mathbf{E}^a}{\partial t}+\frac{1}{\mu_r}\nabla\times\mathbf{B}^a=\mathbf{J}^a. \tag{5.55}$$

The refractive index n is defined in standard dielectric theory as

$$n^2:=\varepsilon_r\mu_r. \tag{5.56}$$

Inserting this into the Faraday law (5.53) gives

$$\frac{\partial \mathbf{B}^a}{\partial t}+n^2\,\nabla\times\mathbf{E}^a=\mathbf{0} \tag{5.57}$$

which is a law of optics. Thus, optical properties can also be described by the ECE polarization and magnetization laws.

5.2.2 Derivation from ECE homogeneous current

Instead of assuming an isotropic, linear medium, we can base our derivation on the more general laws for polarization and magnetization (5.38) and (5.39) directly. Since these are vector laws, we can transfer the ECE polarization index a to $\mathbf{P}$ and $\mathbf{M}$:

$$\mathbf{D}^a=\varepsilon_0\mathbf{E}^a+\mathbf{P}^a, \tag{5.58}$$

$$\mathbf{H}^a=\frac{1}{\mu_0}\mathbf{B}^a-\mathbf{M}^a. \tag{5.59}$$

We insert $\mathbf{D}^a$ and $\mathbf{H}^a$ into the Faraday law in vacuum (5.47), introducing the changes of this law by polarization and magnetization:

$$\frac{1}{c^2}\frac{\partial(\frac{1}{\mu_0}\mathbf{B}^a-\mathbf{M}^a)}{\partial t}+\nabla\times(\varepsilon_0\mathbf{E}^a+\mathbf{P}^a)=\mathbf{0}. \tag{5.60}$$

Rearranging the terms gives

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = \mu_0 \left(\frac{\partial \mathbf{M}^a}{\partial t} - c^2 \nabla \times \mathbf{P}^a \right). \tag{5.61}$$

This is the Faraday law of ECE theory with homogeneous current $\mathbf{j}^a$:

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = \mu_0 \mathbf{j}^a \tag{5.62}$$

with

$$\mathbf{j}^a = \frac{\partial \mathbf{M}^a}{\partial t} - c^2 \nabla \times \mathbf{P}^a. \tag{5.63}$$

In this particular approximation it is seen clearly that the homogeneous current is equivalent to a spacetime with polarization and magnetization. The effect is like a current of magnetic charges, which may be observable only in cosmic dimensions, where it can be amplified by the path length to measurable levels.

■ **Example 5.2** We show that the cosmological red shift can be described by optical properties of spacetime. We cite from [32]:

The homogeneous current (5.63) may appear in cosmic dimensions and is the mechanism responsible for the interaction of gravitation with the light beam as the latter travels from source to telescope, a distance Z. Over this immense distance it is certain that the light beam encounters myriad species of gravitational field before reaching the telescope and the observer. However weak these fields may be in inter-stellar and inter-galactic ECE spacetime, the enormous path length Z amplifies the current $\mathbf{j}^a$ *to measurable levels, and appears in the telescope as a red shift. This inference is analogous to the well known fact that the absorption coefficient in spectroscopy depends on the path length - the greater the path length the greater the absorption of the light beam and the weaker the signal at the detector. Therefore, what is always observed in astronomy, is the effect of gravitation on light through the current of Eq. (5.63) - in general an absorption (or dielectric loss) accompanied by a dispersion (a change in the refractive index).*

It is also well known in spectroscopy that the more dilute the sample the sharper are the spectral features (the effect of collisional broadening is decreased by dilution). Since inter-stellar and inter-galactic spacetime is very tenuous (or dilute), the stars and galaxies appear sharply defined. This does not mean at all that the spacetime is empty or void as in Big Bang theory. The empty inter-stellar and inter-galactic spacetime of Big Bang is defined by Einstein-Hilbert theory alone, without any classical consideration of the classical effect of gravitation on a light beam. The red shifts are defined in Big Bang by a particular solution to the Einstein-Hilbert field equations using a given metric. No account is taken of the homogeneous current $\mathbf{j}^a$ *and so the effect of gravitation on light is not considered classically. These are major omissions, leading to the apparent conclusion that the universe is expanding - simply because the metric demands this conclusion. This is, however, a circular argument - the conclusion (expanding metric deduced) is programmed in at the beginning (expanding metric assumed).*

It is to be stressed that this explanation strongly supports the tired light theory [33]. The common argument against this theory is that light is dispersed by myriads of collisions with particles of the interstellar medium or even quantum vacuum. This should lead to very diffuse images in telescopes. However, according to the explanation above, dispersion does not appear because the inter-stellar medium is very dilute. The red shift is fully explainable on a macroscopic level. The Faraday law can be written for a permeability and permittivity of spacetime, which is space- (and possibly time-) dependent, in the form:

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{B}^a}{\mu_r} \right) + \nabla \times (\varepsilon_r \mathbf{E}^a) = \mathbf{0}. \tag{5.64}$$

Thus, regions with varying μ_r and ε_r alter the electromagnetic properties of light. The red shift is made plausible in the following way: Assume an electromagnetic plane wave, given in cartesian coordinates (with basis $\mathbf{i},\mathbf{j},\mathbf{k}$) by

$$\mathbf{E}^a = \mathbf{i}E_0 e^{i\phi}, \quad \mathbf{B}^a = \mathbf{j}B_0 e^{i\phi} \tag{5.65}$$

with phase factor

$$\phi = \omega t - \kappa Z. \tag{5.66}$$

This is a wave propagating in the $\mathbf{k}$ direction with time frequency ω and wave number $\kappa_Z = \kappa$. We insert this into the Faraday equation of free space (5.46):

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = \mathbf{0} \tag{5.67}$$

and obtain

$$\frac{\partial \mathbf{B}^a}{\partial t} = i\omega \mathbf{j} B_0 e^{i\phi}, \tag{5.68}$$

$$\nabla \times \mathbf{E}^a = -i\kappa \mathbf{j} E_0 e^{i\phi}, \tag{5.69}$$

(see computer algebra code [61] for details). Therefore, we obtain from the Faraday law:

$$\omega B_0 - \kappa E_0 = 0. \tag{5.70}$$

In free space, without dispersion, is

$$\frac{\omega}{\kappa} = c \tag{5.71}$$

and from Section 4.2.3 (the definition of the electromagnetic field tensor) we know that

$$\frac{E_0}{B_0} = c. \tag{5.72}$$

Therefore, the left-hand side of (5.67) gives $1-1=0$, and the Faraday equation is fulfilled. Now we want to know how we have to modify the definitions of the electric and magnetic fields so that the Faraday equation for dielectric space (5.53) is fulfilled:

$$\frac{1}{\mu_r}\frac{\partial \mathbf{B}^a}{\partial t} + \varepsilon_r \nabla \times \mathbf{E}^a = \mathbf{0}. \tag{5.73}$$

Inserting the fields (5.65) leads to

$$\frac{\omega}{\mu_r} B_0 - \varepsilon_r \kappa E_0 = 0. \tag{5.74}$$

Obviously, this equation comes out if we change the definition in the phase factor in the form:

$$\omega \to \frac{\omega}{\mu_r}, \tag{5.75}$$

$$\kappa \to \varepsilon_r \kappa, \tag{5.76}$$

leading to

$$\phi = \frac{\omega}{\mu_r} t - \varepsilon_r \kappa Z. \tag{5.77}$$

(see computer algebra code [61]). The frequency is lowered by a factor of $1/\mu_r$ with $\mu_r > 1$, and this is the cosmological red shift. As explained above, this is an optical effect that has nothing to do with an expanding universe.

From Eq. (5.74) it follows that

$$\frac{\omega}{\kappa} - \varepsilon_r \mu_r c = 0 \tag{5.78}$$

or

$$\frac{\omega}{\kappa} = n^2 c \tag{5.79}$$

where

$$n^2 = \varepsilon_r \mu_r \tag{5.80}$$

is the optical refraction index. In optics, n can be complex valued,

$$n = n' + i\, n'' \tag{5.81}$$

with real part n' and imaginary part n'', describing absorption effects. Then, the frequency value in Eq. (5.79) becomes complex:

$$\omega = n^2 \omega_0 \tag{5.82}$$

where $\omega_0 = \kappa c$ is the frequency of the wave in vacuo. The frequency part of the phase factor becomes

$$e^{i\omega t} = e^{i\, n^2 \omega_0\, t} = e^{i(\omega_r + i\omega_i)t} \tag{5.83}$$

with real and imaginary frequency parts

$$\omega_r = (n'^2 - n''^2)\omega_0, \quad \omega_i = 2n'n''\omega_0. \tag{5.84}$$

This gives two phase factors

$$e^{i\omega t} = e^{i(n'^2 - n''^2)\omega_0\, t}\; e^{-2n'n''\omega_0\, t}. \tag{5.85}$$

The first factor describes a frequency reduction, the second factor an exponential damping of the wave. Many more details are given in [32]. We obtain a light wave that transfers energy to spacetime, resulting in a lowering of the frequency. This is a red shift effect again. ∎

5.3 Conservation theorems

5.3.1 The Pointing theorem

In ECE theory, the Pointing theorem can be developed in the same way as in classical electrodynamics. In addition, a coupling between electromagnetism and gravitation can be included, which is more complete than in the classical theory, because gravitation is described much more completely, compared to classical mechanics [34]. Here we restrict ourselves to the basic features of ECE electrodynamics.

The total rate P of work in a volume V is given by

$$P = \int_V \mathbf{J} \cdot \mathbf{E}\, d^3x. \tag{5.86}$$

Since in ECE theory all electromagnetic quantities have a polarization index a, we can write this directly as

$$P^a = \int_V \mathbf{J}^a \cdot \mathbf{E}^a \, d^3x. \tag{5.87}$$

All subsequent derivations follow the same path as in standard text books [35], except that there is an additional polarization index for all quantities. The energy density of the force fields in materials is

$$u^a = \frac{1}{2}\left(\mathbf{E}^a \cdot \mathbf{D}^a + \mathbf{B}^a \cdot \mathbf{H}^a\right). \tag{5.88}$$

We obtain the following from Eq. (5.86), by substituting terms using the field equations:

$$\boxed{\frac{\partial u^a}{\partial t} + \nabla \cdot \mathbf{S}^a = -\mathbf{J}^a \cdot \mathbf{E}^a.} \tag{5.89}$$

This is the *Poynting theorem*, in which the *Poynting vector* $\mathbf{S}^a$ is defined by

$$\boxed{\mathbf{S}^a := \mathbf{E}^a \times \mathbf{H}^a.} \tag{5.90}$$

The Poynting vector has the dimensions of energy/(area·time) and describes the energy flow of the fields. The term $\mathbf{J}^a \cdot \mathbf{E}^a$ is the energy density originating in charges moving in an electric field. Magnetic fields are irrelevant since the charges move perpendicularly to the magnetic field due to the Lorentz force. The Poynting vector describes the energy flow and is proportional to the electromagnetic momentum.

In standard theory, only the energy density and energy flow that originate from the force fields are considered. In ECE theory, the potential also is physical. Spacetime itself can be considered as a background potential. Therefore, a potential without fields is a kind of flux field and has a field energy. This type of potential is not taken into account in the Poynting theorem. There is no classical counterpart for the background potential. There are some possible approaches in [36], for example, for the vector and scalar potential of ECE theory:

$$u_{A_{ECE}}(\mathbf{r},t) = \frac{1}{2\mu_0}\sum_i \left(\frac{1}{c^2}\left|\omega_0 A_i\right|^2 + \left|\omega_i A_i\right|^2\right), \tag{5.91}$$

$$u_{\Phi_{ECE}}(\mathbf{r},t) = \frac{1}{2}\varepsilon_0 \left(\frac{1}{c^2}(\omega_0 \Phi)^2 + \sum_i \left|\omega_i \Phi\right|^2\right). \tag{5.92}$$

The i index numbers the components of the vectors $\mathbf{A}$ and ω.

5.3.2 The continuity equation

In electrodynamics, charges and current densities are conserved. In ECE theory, both are geometrical quantities, which are subject to change due to the structure of spacetime. Nevertheless, they are conserved as in classical theory, indicating that the ECE approach is in agreement with essential physics concepts. The second field equation (4.48) reads

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu}, \tag{5.93}$$

where $J^{a\nu}$ is the ECE 4-current density as given by Eq. (4.50). According to (4.172), the electromagnetic field tensor is rewritten to contravariant form:

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + \omega^{a\mu}{}_b A^{b\nu} - \omega^{a\nu}{}_b A^{b\mu}. \tag{5.94}$$

The derivative in (5.93) can therefore be written:

$$\partial_\mu F^{a\mu\nu} = \partial_\mu \partial^\mu A^{a\nu} - \partial_\mu \partial^\nu A^{a\mu} + \partial_\mu (\omega^{a\mu}{}_b A^{b\nu}) - \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}). \tag{5.95}$$

We apply an additional derivative ∂_ν to both sides of (5.93). The left-hand side then becomes

$$\partial_\nu \partial_\mu F^{a\mu\nu} = \partial_\nu \partial_\mu \partial^\mu A^{a\nu} - \partial_\nu \partial_\mu \partial^\nu A^{a\mu} + \partial_\nu \partial_\mu (\omega^{a\mu}{}_b A^{b\nu}) - \partial_\nu \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}). \tag{5.96}$$

μ and ν are dummy indices, their names can be interchanged. Partial derivatives can also be commuted:

$$\partial_\nu \partial_\mu F^{a\mu\nu} = \partial_\nu (\partial_\mu \partial^\mu) A^{a\nu} - \partial_\nu (\partial_\mu \partial^\mu) A^{a\nu} + \partial_\nu \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}) - \partial_\nu \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}). \tag{5.97}$$

The terms at the rigut-hand side of the last equation cancel out, resulting in

$$\partial_\nu \partial_\mu F^{a\mu\nu} = 0. \tag{5.98}$$

Inserting this into Eq. (5.93) gives the *continuity equation* in generally covariant form:

$$\boxed{\partial_\nu J^{a\nu} = 0,} \tag{5.99}$$

which is the 4-divergence of the 4-current density. By applying Eq. (4.53), $J^{a0} = c\rho^a$, this can be written in vector form:

$$\boxed{\frac{\partial \rho^a}{\partial t} + \nabla \cdot \mathbf{J}^a = 0.} \tag{5.100}$$

This form of the continuity equation is identical to that of standard electrodynamics, but holds in a spacetime with curvature and torsion. Thus, the range of validity has been expanded significantly.

5.4 Examples of ECE electrodynamics

5.4.1 Gravity-induced polarization changes

■ **Example 5.3** As shown in Example 5.2, the electromagnetic fields of spacetime have optical properties, leading to magnetization and polarization. Here we apply this to polarization changes, which are induced by gravity. Assume that a circularly polarized electromagnetic wave travels through space in the Z direction. We assume only one polarization of the a index, therefore the index can be omitted. According to Eqs. (4.150, 4.151) of Example 4.2, the electric and magnetic field (induction) of the wave are then:

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}(\mathbf{i} + i\mathbf{j})e^{i\phi}, \tag{5.101}$$

$$\mathbf{B} = \frac{B^{(0)}}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})e^{-i\phi} \tag{5.102}$$

with phase factor

$$\phi = \omega t - \kappa Z. \tag{5.103}$$

In an optically active region of spacetime with $\mu_r \neq 1$ and $\varepsilon_r \neq 1$ the phase is changed to

$$\phi_1 = \frac{\omega}{\mu_r} t - \varepsilon_r \kappa Z. \tag{5.104}$$

With the definition of the refraction index

$$n^2 = \mu_r \varepsilon_r \tag{5.105}$$

the Faraday law in media, Eq. (5.73), then reads

$$\frac{1}{n}\frac{\partial \mathbf{B}^a}{\partial t} + n\nabla \times \mathbf{E}^a = \mathbf{0}. \tag{5.106}$$

The force fields are changed according to

$$\mathbf{E} \to n\mathbf{E}, \qquad \mathbf{B} \to \frac{1}{n}\mathbf{B}. \tag{5.107}$$

The real and physical part of Eq. (5.101) in vacuo is

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}\left(\mathbf{i}\cos(\phi) + \mathbf{j}\sin(\phi)\right) \tag{5.108}$$

(see computer algebra code [62]). In an optically active spacetime the phase factor ϕ is modified to ϕ_1 as described above. Then:

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}\left(\mathbf{i}\cos(\phi_1) + \mathbf{j}\sin(\phi_1)\right). \tag{5.109}$$

Since $\mathbf{E}$ depends on the phase factor in a nonlinear way, the ratio between the X and Y components changes. If

$$\cos(\phi_1) = a\cos(\phi), \tag{5.110}$$
$$\sin(\phi_1) = b\sin(\phi), \tag{5.111}$$

we then have

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}\left(a\,\mathbf{i}\cos(\phi) + b\,\mathbf{j}\sin(\phi)\right). \tag{5.112}$$

This is an elliptically polarized wave. For example, for $\phi = 45°$, $\phi_1 = 60°$, we obtain the values $a = 0.707$, $b = 1.225$ (see computer algebra code [62]).

It has been shown that changes of the optical properties of spacetime, due to matter, effect a change of polarization in light passing through a region where these properties are affected by gravitation. Such polarization changes from a white dwarf have been reported by Preuss et al. [37]. Details are discussed in [38]. The ECE theory for this example describes the change of polarization qualitatively and straightforwardly, and a quantitative description could be developed for given parameter functions $\mu_r(\mathbf{r})$ and $\varepsilon_r(\mathbf{r})$ or, alternatively, for given curvature/torsion parameters of the homogeneous current in the respective region. This effect is not present in Einsteinian general relativity, so ECE is a preferred theory. ∎

5.4.2 Effects of spacetime properties on optics and spectroscopy

∎ **Example 5.4** We show that the Sagnac effect is a consequence of rotating spacetime [39]. Consider the rotation of a beam of light of any polarization around a circle in the XY plane at an angular frequency ω_1 to be determined. The rotation is a rotation of spacetime described by the rotating tetrad field vector

$$\mathbf{q}^{(1)} = \frac{1}{\sqrt{2}}\left(\mathbf{i} - i\mathbf{j}\right)e^{i\omega_1 t}, \tag{5.113}$$

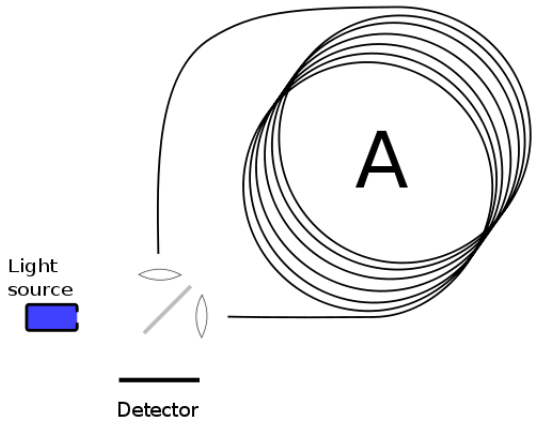

Figure 5.1: Sagnac interferometer [70].

i.e., rotation around the rim of the circular platform of the static Sagnac interferometer with the beam of light, see Fig. 5.1. The ECE ansatz converts the geometry into physics as follows:

$$\mathbf{A}^{(1)} = A^{(0)}\mathbf{q}^{(1)}. \tag{5.114}$$

This equation describes a vector potential field rotating around the rim of the circular Sagnac platform at rest. Rotation to the left is described by:

$$\mathbf{A}_L^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})\, e^{i\omega_1 t}, \tag{5.115}$$

and rotation to the right by:

$$\mathbf{A}_R^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} + i\mathbf{j})\, e^{i\omega_1 t}. \tag{5.116}$$

This can be seen by computing the real and physical parts:

$$Re(\mathbf{A}_L^{(1)}) = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i}\cos(\omega_1 t) + \mathbf{j}\sin(\omega_1 t)), \tag{5.117}$$

$$Re(\mathbf{A}_R^{(1)}) = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i}\cos(\omega_1 t) - \mathbf{j}\sin(\omega_1 t)), \tag{5.118}$$

which are circular motions in the left and right directions (see computer algebra code [63]).

When the platform is at rest, a beam going around left-wise takes the same time to reach its starting point on the circle as a beam going around right-wise. The time delay between the two beams is:

$$\Delta t = 2\pi\left(\frac{1}{\omega_1} - \frac{1}{\omega_1}\right) = 0. \tag{5.119}$$

Note carefully that Eqs. (5.115) and (5.116) do not exist in special relativity because electromagnetism is thought of as an entity superimposed on a passive or static frame which never rotates. Now consider the beam (5.115) rotating left-wise and spin the platform left-wise at an angular frequency Ω. The result is an increase in the angular frequency of the rotating tetrad, (because the spacetime is spinning more quickly):

$$\omega_1 \to \omega_1 + \Omega. \tag{5.120}$$

Similarly, consider the beam (5.115) rotating left-wise and spin the platform right-wise at the same angular frequency Ω. The result is a decrease in the angular frequency of the rotating tetrad (because the spacetime is spinning more slowly):

$$\omega_1 \rightarrow \omega_1 - \Omega. \tag{5.121}$$

The time delay between a beam circling left-wise with the platform and a beam circling left wise against the platform is:

$$\Delta t = 2\pi \left(\frac{1}{\omega_1 - \Omega} - \frac{1}{\omega_1 + \Omega} \right) = \frac{4\pi\Omega}{\omega_1^2 - \Omega^2} \tag{5.122}$$

and this is the *Sagnac effect*. Winding more turns of fiber on the interferometer, as indicated in Fig. 5.1, increases the time difference as a multiple of the number of windings.

In order to calculate the angular frequency ω_1 of the rotating light, we start with the fact that the time it takes for light to traverse an infinitesimal length element dl is

$$dt = \frac{dl}{c}. \tag{5.123}$$

The apparatus rotates in this time by an angle Ωdt, and the radius of the interferometer is r. Then the tangential velocity v of mechanical rotation at radius r is $v = \Omega r$ (in the case $v << c$). The amount of increase or decrease in the path length of the beam, in a tangential direction, is

$$dx = \Omega\, r\, dt = \frac{\Omega\, r}{c} dl. \tag{5.124}$$

For a complete rotation we obtain

$$x = \oint dx = \oint \frac{\Omega\, r}{c} dl = \frac{\Omega}{c} \cdot 2\pi r \cdot r = \frac{\Omega}{c} \cdot 2A, \tag{5.125}$$

where $A = \pi r^2$ is the area enclosed by a circular beam. The difference between the paths of both circulating light beams is $2x$, therefore from (5.123):

$$\Delta t = \frac{2x}{c} = \frac{4\Omega}{c^2} A \tag{5.126}$$

and equating this with (5.122):

$$\frac{4\Omega}{c^2} \pi r^2 = \frac{4\pi\Omega}{\omega_1^2 - \Omega^2}. \tag{5.127}$$

For $\Omega << \omega_1$ we obtain (see computer algebra code [63]):

$$\omega_1 = \frac{c}{r} = c\kappa \tag{5.128}$$

with a wave number $\kappa = 1/r$. This is the angular frequency of the rotating tetrad, or rotating spacetime. ∎

The Sagnac effect is an example of a geometrical phase effect, which is also called a Berry phase. In quantum physics, the phase of the wave function changes when a quantum mechanical object is moved on different paths from point A to point B, or from B back to point A. Prominent examples are the Aharonov-Bohm effect (see later in this book) and the Tomita-Chiao effect. The latter has been explained by ECE theory in [40]. The essential result of this effect is that light in an optical fiber changes its phase, when the fiber is laid straight or in a curved way. This is not due to differences in refraction when the fiber is bent or wound around a cylinder. In standard physics, a Berry phase is explained by complicated quantum mechanical methods, but ECE theory is able to give explanations on the classical level, as we did for the Sagnac interferometer.

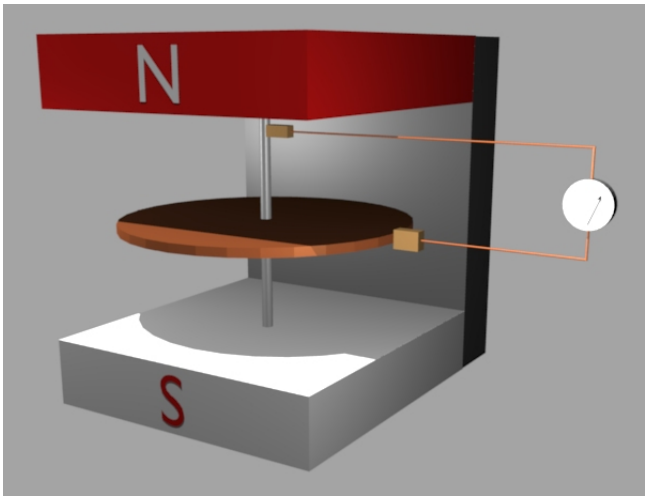

Figure 5.2: Principle of the homopolar generator (Faraday disk) [71].

5.4.3 The homopolar generator

■ **Example 5.5** The homopolar generator or Faraday disk is the first electric generator, invented by Faraday in 1831. The original experiment by Faraday was recorded in his diary on Dec 26$^{\text{th}}$ 1831, and consisted of a disc placed on top of a permanent magnet and separated from the magnet by paper. The assembly was suspended by a string and the complete assembly rotated. An electro-motive force (electric field) was observed between the center of the disc and the outer edge of the disc. The electro-motive force vanished when the mechanical torsion (rotation) was absent.

The Faraday law of induction of the standard model (special relativistic electrodynamics) later emerged to describe the induction seen when a magnet is translated with respect to a stationary induction loop. This law does not cover the Faraday disk generator, in which the magnet is stationary. In standard electrodynamics, the Faraday disk is explained by the Lorentz force law, which is the translation law of charges moving in a magnetic field. Since the Lorentz force law is not part of the Maxwell-Heaviside equations, it is sometimes stated that the homopolar generator is not explainable by standard electrodynamics, although this point of view is more or less arbitrary.

In standard theory, any field is considered to be an entity distinct from the passive frame, especially if the field is moving or spinning. When the Faraday disc is described by ECE theory, the frame itself is spinning. This can be described by the circular complex basis as shown, for example, in Eq. (5.113). The two transversal basis vectors $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ can be described by

$$\mathbf{q}^{(1)} = \mathbf{q}^{(2)*} = \frac{1}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})\, e^{i\Omega t}. \tag{5.129}$$

Ω is the frequency with which the disc is mechanically spun. According to the Evans ansatz (5.114), this generates vector potentials

$$\mathbf{A}^{(1)} = A^{(0)}\mathbf{q}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})\, e^{i\Omega t}, \tag{5.130}$$

$$\mathbf{A}^{(2)} = \mathbf{A}^{(1)*} = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} + i\mathbf{j})\, e^{-i\Omega t}. \tag{5.131}$$

Both vector potentials have the same real and physical part:

$$Re(\mathbf{A}^{(1)}) = Re(\mathbf{A}^{(2)}) = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i}\cos(\Omega t) + \mathbf{j}\sin(\Omega t)). \tag{5.132}$$

According to Eq. (4.197), the ECE electric field is

$$\mathbf{E}^a = -\nabla\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{A}^b + \omega^a{}_b\phi^b. \tag{5.133}$$

Since the electric potential ϕ and the spin connections are zero (there is no a priori given electric structure), the electric field evoked by mechanical rotation is

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t}. \tag{5.134}$$

With (5.130, 5.131) this leads to the electric fields

$$\mathbf{E}^{(1)} = \frac{A^{(0)}\Omega}{\sqrt{2}}(-i\mathbf{i} - \mathbf{j})\, e^{i\Omega t}, \tag{5.135}$$

$$\mathbf{E}^{(2)} = \frac{A^{(0)}\Omega}{\sqrt{2}}(i\mathbf{i} - \mathbf{j})\, e^{-i\Omega t}, \tag{5.136}$$

whose real part is

$$\mathbf{E} = Re(\mathbf{E}^{(1)}) = Re(\mathbf{E}^{(2)}) = \frac{A^{(0)}\Omega}{\sqrt{2}}\left(\mathbf{i}\sin(\Omega t) - \mathbf{j}\cos(\Omega t)\right), \tag{5.137}$$

(see computer algebra code [64]). This electric field (with strength in volts per meter) spins around the rim of the rotating disk. As observed experimentally, it is proportional to the product of $A^{(0)}$ and Ω, and the factor $A^{(0)}$ stems from the permanent magnet. An electromotive force is set up between the center of the disk and its rim, as first observed by Faraday, and this quantity is measured by a voltmeter at rest with respect to the spinning disk.

The frequency Ω of mechanical rotation can be considered as a spin connection. Then Eq. (5.133) can be written as

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t} - \Omega\mathbf{A}^a. \tag{5.138}$$

The real part of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ contains sin and cos terms, but gives a graph equivalent to (5.137) (see computer algebra code [64]). In [41, 42] the spin connection was defined with a complex phase factor:

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t} - i\Omega\mathbf{A}^a. \tag{5.139}$$

This gives the simpler result:

$$\mathbf{E} = Re(\mathbf{E}^{(1)}) = 2\frac{A^{(0)}\Omega}{\sqrt{2}}\left(\mathbf{i}\sin(\Omega t) - \mathbf{j}\cos(\Omega t)\right), \tag{5.140}$$

which is - except for a constant factor - identical to (5.137). ∎

ECE not only gives a consistent description of the Faraday disk through the field equations, but also allows for resonance enhancements of the induced voltage. This is reported in detail in [42]. Typical simple resonance effects are described next.

5.4.4 Spin connection resonance

■ **Example 5.6** We now consider the resonant Coulomb law. One of the most important consequences of general relativity applied to electrodynamics is that the spin connection enters the relation between the field and potential as described in Section 4.4. The equations of electrodynamics, as written in terms of the potential, can be reduced to the form of Euler-Bernoulli resonance equations. The method is most simply illustrated by considering the vector form of the Coulomb law deduced in Section 4.2.3:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{5.141}$$

where we have written the fields without polarization index. Assuming the absence of a vector potential (absence of a magnetic field), the electric field in the standard model is

$$\mathbf{E} = -\nabla\phi, \tag{5.142}$$

where ϕ is the electric potential. Under the same assumption, the electric field in ECE theory, according to Eq. (4.197), is

$$\mathbf{E} = -\nabla\phi + \omega\phi, \tag{5.143}$$

where ω is the vector spin connection. Therefore, Eq. (5.141) takes on the form

$$\nabla^2\phi - \omega \cdot \nabla\phi - (\nabla \cdot \omega)\,\phi = -\frac{\rho}{\varepsilon_0}. \tag{5.144}$$

The equivalent equation in the standard model is the Poisson equation, which is a limit of Eq. (5.144) when the spin connection is zero. The Poisson equation does not give resonant solutions. However, Eq. (5.144) has resonant solutions of Euler-Bernoulli type, as can be seen in the following discussion. Restricting consideration to one cartesian coordinate, we have only the dependencies $\phi(X)$ and the spin connection has only an X component $\omega_X(X)$. Then Eq. (5.144) reads:

$$\frac{d^2\phi}{dX^2} - \omega_X\frac{d\phi}{dX} - \frac{d\omega_X}{dX}\phi = -\frac{\rho}{\varepsilon_0}. \tag{5.145}$$

This equation has the structure of a damped Euler-Bernoulli resonance of the form

$$\frac{d^2\phi}{dx^2} + \alpha\frac{d\phi}{dx} + \kappa_0^2\phi = F_0\cos(\kappa x), \tag{5.146}$$

if we assume $\omega_X < 0$. Below we will see that this is not a real restriction. Here, κ_0 is the spatial eigenfrequency, measured in 1/m, and α is the damping constant. At the right-hand side, there is a periodic driving force with spatial frequency (wave number) κ. The particular solution of this differential equation is

$$\phi = F_0\frac{\alpha\kappa\sin(\kappa x) + \left({\kappa_0}^2 - \kappa^2\right)\cos(\kappa x)}{(\kappa_0^2 - \kappa^2)^2 + \alpha^2\kappa^2}. \tag{5.147}$$

For vanishing damping, we have

$$\phi \to F_0\frac{\cos(\kappa x)}{(\kappa_0^2 - \kappa^2)^2}. \tag{5.148}$$

For $\kappa \to \kappa_0$ the amplitude of $\phi(x)$ approaches infinity. In the case of damping, the amplitude in the resonance point remains finite (see examples in Fig. 5.3).

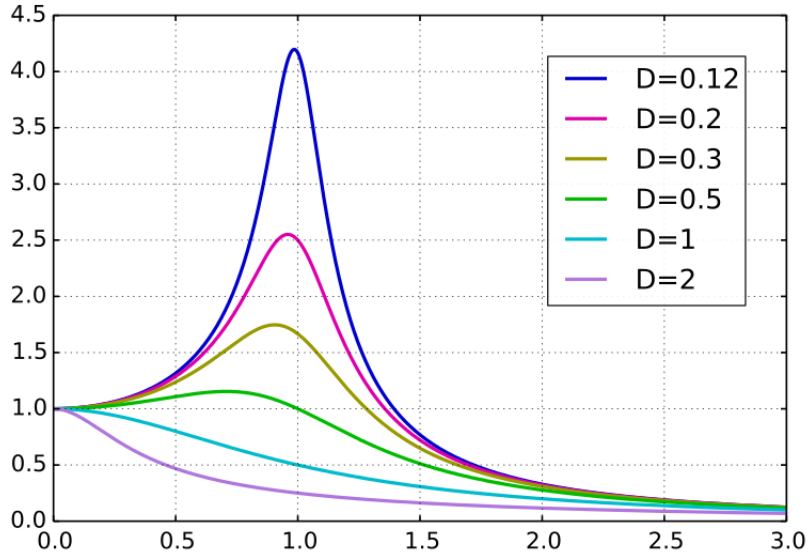

Figure 5.3: Y axis: steady-state amplitude $\phi/\phi_{\text{static}}$ of a damped driven oscillator with different damping constants $D = \alpha/2$. X axis: frequency ratio κ/κ_0 [72].

By comparing Eqs. (5.145) and (5.146), it is seen that the Coulomb equation (5.144) has no constant coefficients and thus is not an original form of the Euler-Bernoulli resonance. Therefore, we can expect that the solutions may differ significantly from those of the original Euler-Bernoulli equation. To investigate this, we consider an example in spherical polar coordinates. We assume that the potential and the spin connection depend only on the radial coordinate r. For the radial (and only) component of the spin connection we assume

$$\omega_r = \frac{1}{r}. \tag{5.149}$$

The differential operators in (5.144) then take the form

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}, \tag{5.150}$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\cdot\mathbf{e}_r, \tag{5.151}$$

$$\nabla\cdot(\omega_r\,\mathbf{e}_r) = -\frac{1}{r^2}. \tag{5.152}$$

Then Eq. (5.144), with the right-hand side replaced by an oscillatory driving term, reads:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\phi}{r^2} = F_0\sin(\kappa r). \tag{5.153}$$

This equation can be solved analytically (see computer algebra code [65]). The solution contains an expression $-\cos(\kappa r)/r$, leading to the limit $\phi(r) \to -\infty$ in the case $r \to 0$, as graphed in Fig. 5.4. Other model examples for ω are listed in the code. Although the Coulomb law with the spin connection term resembles a resonance equation with damping, there is no damping for $r \to 0$ because of the non-constant coefficients in the equation. ∎

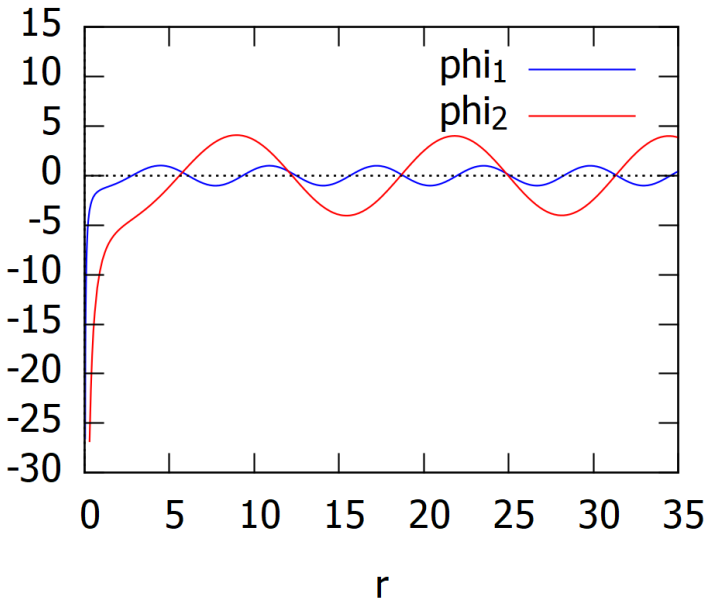

Figure 5.4: Solution of Eq. (5.153), $\kappa = 1$ and $\kappa = 0.5$, other constants normalized.

The spin connection has already been incorporated during the course of development of ECE theory into the Coulomb law, which is the basic law used in the development of quantum chemistry. This process has been illustrated with the hydrogen atom [43]. It serves as a model system for the huge class of atomic, molecular and solid-state physics. (The most used method for computation of electronic properties of solids is Density Functional Theory.) We will come back to this in the quantum physics part of this book.

The ECE theory has also been used to design or explain circuits, which use spin connection resonance to take power from spacetime, notably in papers 63 and 94 of the ECE series on www.aias.us [43, 44]. In paper 63, the spin connection was incorporated into the Coulomb law and the resulting equation in the scalar potential shown to have resonance solutions using an Euler transform method. In paper 94, this method was extended and applied systematically to the Bedini machine, which was shown to have the chance of producing energy from spacetime, although nobody has succeeded in achieving this to date. In addition, spacetime effects in transformers have been found by Ide and successfully explained by ECE theory [45].

■ **Example 5.7** As another important example, we consider resonant forms of the Ampère-Maxwell law. In potential representation, see Eq. (4.206), it reads

$$\begin{aligned}&\nabla\left(\nabla\cdot\mathbf{A}^a\right)-\nabla^2\mathbf{A}^a-\nabla\times\left(\omega^a{}_b\times\mathbf{A}^b\right)\\&+\frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}^a}{\partial t^2}+c\frac{\partial\left(\omega^a{}_{0b}\mathbf{A}^b\right)}{\partial t}+\nabla\frac{\partial\phi^a}{\partial t}-\frac{\partial\left(\omega^a{}_b\phi^b\right)}{\partial t}\right)=\mu_0\mathbf{J}^a.\end{aligned}\tag{5.154}$$

$\mathbf{J}^a$ is a current, which may have a polarization dependence. Assuming the simple case that there is no scalar potential, and that the vector potential is independent of space location and has a pure time dependence only, we obtain the equation

$$\frac{\partial^2\mathbf{A}^a}{\partial t^2}+c\frac{\partial\left(\omega^a{}_{0b}\mathbf{A}^b\right)}{\partial t}=\frac{1}{\varepsilon_0}\mathbf{J}^a.\tag{5.155}$$

Restricting this equation to one polarization index, we have

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} + c\frac{\partial\left(\omega_0 \mathbf{A}\right)}{\partial t} = \frac{1}{\varepsilon_0}\mathbf{J}. \tag{5.156}$$

This equation is formally identical to (5.145), except that it is a vector equation and the (only) coordinate is the time coordinate. Here, the spin connection is the scalar spin connection ω_0 with units of 1/m. We replace it by a time frequency, subsuming the factor c:

$$\omega_t = c\ \omega_0 \tag{5.157}$$

so that (5.156) can be written:

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} + \omega_t \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \omega_t}{\partial t}\mathbf{A} = \frac{1}{\varepsilon_0}\mathbf{J}, \tag{5.158}$$

in full analogy to (5.145). Therefore, the existence of the time spin connection makes the Ampère-Maxwell law a resonance equation in the same way as discussed for the Coulomb law in the preceding example.

Another resonance is possible, when we assume that the vector potential is only space-dependent and there is no scalar potential, for example in magnetic structures. If $\mathbf{A}$ is divergence-free, we obtain from Eq. (5.154), again for one direction of polarization:

$$-\nabla^2 \mathbf{A} - \nabla \times (\boldsymbol{\omega} \times \mathbf{A}) = \mu_0 \mathbf{J}. \tag{5.159}$$

Here the vector spin connection $\boldsymbol{\omega}$ appears again. Using the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \tag{5.160}$$

and that $\mathbf{A}$ is divergence-free, we obtain

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{A}) = -\mathbf{A}(\nabla \cdot \boldsymbol{\omega}) + (\mathbf{A} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{A} \tag{5.161}$$

so that Eq. (5.159) becomes

$$\nabla^2 \mathbf{A} + \mathbf{A}(\nabla \cdot \boldsymbol{\omega}) - (\mathbf{A} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{A} = -\mu_0 \mathbf{J}. \tag{5.162}$$

It can be seen that this equation contains differentiations of $\mathbf{A}$ in zeroth, first and second order. Obviously, resonances are possible for this special form of the Ampère-Maxwell law. To show this, we first define a special system, where $\mathbf{A}$ is restricted to two dimensions and the spin connection is perpendicular to the plane of $\mathbf{A}$. In cartesian coordinates, we then have

$$\mathbf{A} = \begin{bmatrix} A_X \\ A_Y \\ 0 \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega_Z \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} J_X \\ J_Y \\ J_Z \end{bmatrix}, \tag{5.163}$$

where all variables depend on coordinates X and Y. As shown in computer algebra code [66], it follows that

$$\nabla^2 \mathbf{A} = \begin{bmatrix} \frac{\partial^2 A_X}{\partial X^2} + \frac{\partial^2 A_X}{\partial Y^2} \\ \frac{\partial^2 A_Y}{\partial X^2} + \frac{\partial^2 A_Y}{\partial Y^2} \\ 0 \end{bmatrix}, \tag{5.164}$$

$$\boldsymbol{\omega} \times \mathbf{A} = \begin{bmatrix} -A_Y \omega_Z \\ A_X \omega_Z \\ 0 \end{bmatrix}, \tag{5.165}$$

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{A}) = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial \omega_Z}{\partial X} A_X + \frac{\partial \omega_Z}{\partial Y} A_Y + \frac{\partial A_X}{\partial X}\omega_Z + \frac{\partial A_Y}{\partial Y}\omega_Z \end{bmatrix}. \tag{5.166}$$

Inserting this into Eq. (5.159), leads to three component equations:

$$\frac{\partial^2 A_X}{\partial X^2} + \frac{\partial^2 A_X}{\partial Y^2} = -\mu_0 J_X, \tag{5.167}$$

$$\frac{\partial^2 A_Y}{\partial X^2} + \frac{\partial^2 A_Y}{\partial Y^2} = -\mu_0 J_Y, \tag{5.168}$$

$$\frac{\partial \omega_Z}{\partial X} A_X + \frac{\partial \omega_Z}{\partial Y} A_Y + \frac{\partial A_X}{\partial X} \omega_Z + \frac{\partial A_Y}{\partial Y} \omega_Z = -\mu_0 J_Z. \tag{5.169}$$

The first two equations decouple from the third, which is of first order in derivatives only. In order to get an impression of how resonances can occur, we simplify this equation set further, so that only one variable $A_X(X)$ is left:

$$\mathbf{A} = \begin{bmatrix} A_X \\ 0 \\ 0 \end{bmatrix}, \tag{5.170}$$

where ω and $\mathbf{J}$ remain as in (5.163) but depend on the X variable only. Then the equation set (5.167-5.169) simplifies to

$$\frac{\partial^2 A_X}{\partial X^2} = -\mu_0 J_X, \tag{5.171}$$

$$0 = -\mu_0 J_Y, \tag{5.172}$$

$$\frac{\partial \omega_Z}{\partial X} A_X + \frac{\partial A_X}{\partial X} \omega_Z = -\mu_0 J_Z. \tag{5.173}$$

From Eq. (5.172) follows $J_Y = 0$ as a constraint. Eqs. (5.171) and (5.173) are not compatible any more, but we add both equations to obtain an analytically solvable equation that combines the properties of both equations:

$$\frac{\partial^2 A_X}{\partial X^2} + \frac{\partial A_X}{\partial X} \omega_Z + \frac{\partial \omega_Z}{\partial X} A_X = -\mu_0 (J_X + J_Z). \tag{5.174}$$

(Below, we will see that this is a meaningful operation.) This is a resonance equation with non-constant coefficients, as were Eqs. (5.145) and (5.158). For demonstration, we present some solutions for this equation in cartesian coordinates. In Table 5.1 we show four solutions of Eq. (5.174) for given combinations of current density $\mathbf{J}$ and spin connection ω_Z. These are graphed in Fig. 5.5. All solutions have divergence points for $X \to 0$, $X \to \pm\infty$, or elsewhere. Eq. (5.171) has an oscillatory solution as expected, but Eq. (5.173), although only of first order, has diverging solutions (solutions 5-7, see Table 5.1 and Fig. 5.6). Therefore, the combined equation (5.174) is an approximation to a full-blown calculation where all components of $\mathbf{A}$ and ω are present, as in Eq. (5.159). ∎

Equation	Fig. ref.	J_X, J_Z	ω_Z	Solution
(5.174)	solution 1	J_0	$1/X$	$\frac{2}{3}J_0\mu_0 X^2$
	solution 2	J_0/X	$1/X$	$-J_0\mu_0(X\log(X)+\frac{1}{2}X)$
	solution 3	$J_0\sin(aX)$	$1/X$	$2\frac{J_0\mu_0}{a^3}(\sin(aX)+\frac{\cos(aX)}{X})$
	solution 4	J_0X^2	$1/X$	$\frac{2}{15}J_0\mu_0X^4$
(5.171)	solution 5	$J_0\cos(\kappa_0X)$		$\frac{\mu_0}{\kappa_0^2}\cos(\kappa_0X)$
(5.173)	solution 6	$J_0\cos(\kappa_0X)$	$\kappa_0\cos(\kappa X)$	$-X\frac{\mu_0}{\kappa_0^2}\frac{\sin(\kappa_0X)}{\cos(\kappa X)}$
	solution 7	$J_0\cos(\kappa_0X)$	$1/X$	$-X\frac{\mu_0}{\kappa_0}\sin(\kappa_0X)$

Table 5.1: Solutions of model resonance equations.

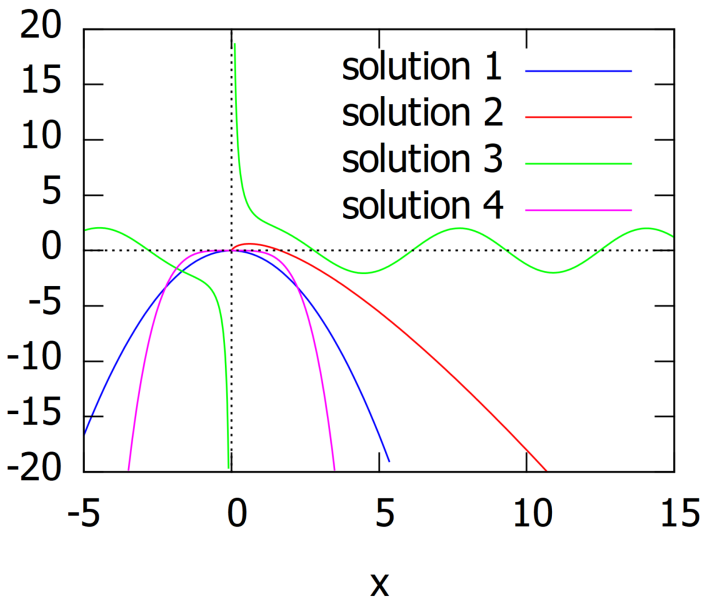

Figure 5.5: Solutions of Eq. (5.174), all constants normalized.

It is known from the work of Tesla, for example, that strong resonances in electric power can be obtained with a suitable apparatus, and such resonances cannot be explained using the standard model. One consistent explanation of Tesla's well-known results is given by the incorporation of the spin connection into classical electrodynamics.

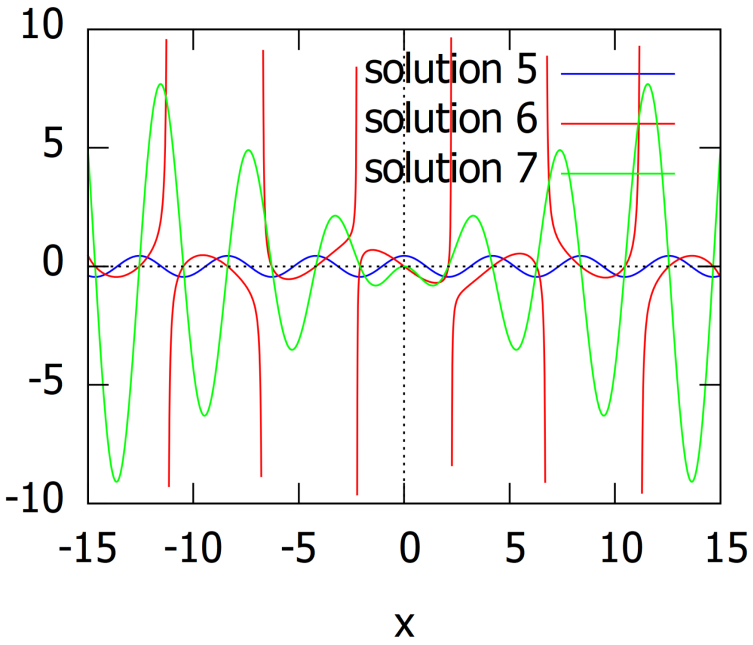

Figure 5.6: Solutions of Eqs. (5.171) and (5.173) with $\kappa_0 = 1.5$ and $\kappa = 0.7$, other constants normalized.

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[45] Papers 364, 382, 383, Unified Field Theory (UFT) Section of www.aias.us.

Computer algebra code (Maxima)

[46] Ex2.4.wxm

[47] Ex2.5.wxm

[48] Ex2.10.wxm

[49] CH02-diag-metric.wxm

[50] CH02-nondiag-metric.wxm

[51] Ex2.11.wxm

[52] Ex2.12.wxm

[53] Ex2.13.wxm

[54] Ex2.14.wxm

[55] Ex2.15.wxm

[56] Ex3.1.wxm

[57] Ex3.2a.wxm

[58] Ex3.2b.wxm

[59] Ex4.1.wxm

[60] Ex4.2.wxm

[61] Ex5.2.wxm

[62] Ex5.3.wxm

[63] Ex5.4.wxm

[64] Ex5.5.wxm

[65] Ex5.6.wxm

[66] Ex5.7.wxm

External figures and pictures

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[72] Wikimedia Commons contributors, "File:Mplwp resonance Dmany.svg," Wikimedia Commons, the free media repository, https://commons.wikimedia.org/w/index.php?title=File:Mplwp_resonance_Dmany.svg&oldid=229436939 (accessed March 5, 2020).

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